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Exploring a Connection between Transformational Geometry and Matrices

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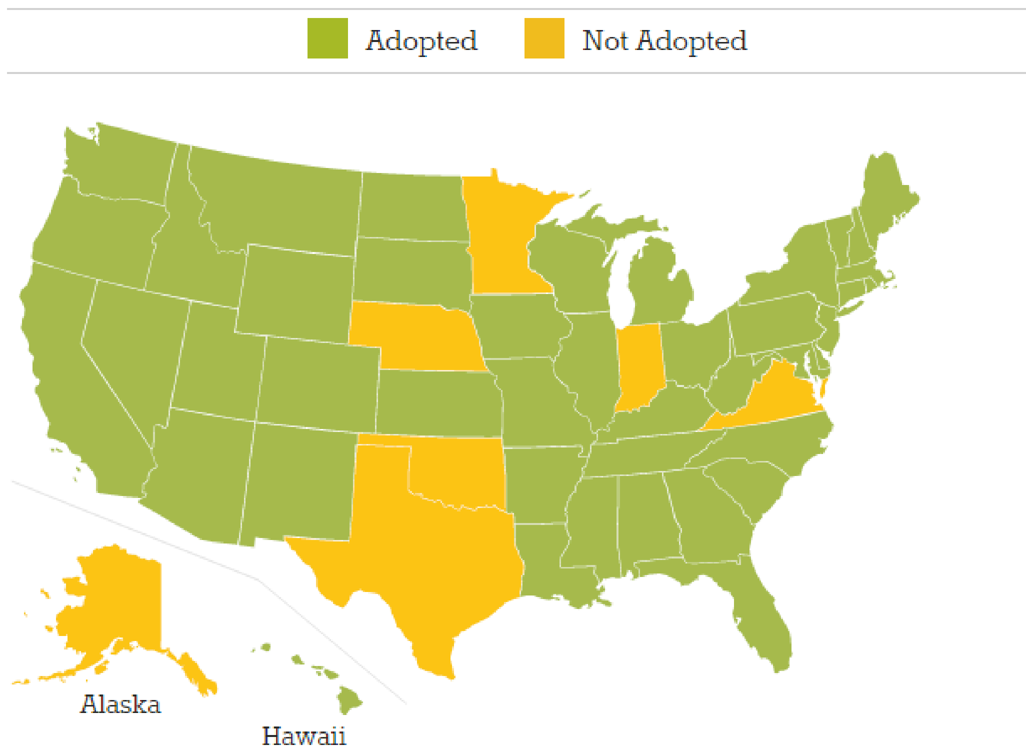
Introduction

The traditional mathematics curriculum in the United States is characterized by the amount topics that are not connected. For example, algebra and geometry are taught in different years with different textbooks. At the college level the story is not different. One of the changes promoted by the Common Core State Standards is the emphasis on connections. As a future mathematics teacher I am interested in ways to connect content. In my work for this honors thesis I was interested in learning more about the natural connections among topics in mathematics. I selected geometry as my focus because of the big changes in this particular area in the Common Core. I explored the topic of transformations, in different ways. First, I learned more about the changes in the geometry curriculum for middle and high school in relation to geometry. Then, from a mathematical point of view, I learned more about the symmetry groups for friezes, a form of strip patterns. I used GeoGebra to help me understand the different transformations that are present in each group. Finally, I had the opportunity to teach a lesson that explicitly makes connections between geometric transformations and matrices. I present the mathematical connection between transformations and matrices and the lesson I taught to help students explore this connection.

The Common Core State Standards

The Common Core State Standards (CCSS) (2010) consists of the learning goals for what students should be able to do and know at each grade level. The standards provide teachers a guide to ensure that their students have gained the knowledge and skills needed to be successful in college or in careers. Common Core Standards are now in place for both Mathematics and English Language Arts for students in kindergarten through 12th grade. The Common Core was developed by building on the standards that were in place in many states, by examining the expectations of other high-performing countries, and from results from research on how students learn. With involvement from different stakeholders, the standards were created to be realistic and practical for the classroom (CCSSI, 2010).

As previous content standards varied across the states, one of the goals of the Common Core is to create high standards that are consistent across the states. This also contributes to collaboration across the states in development of teaching materials, development and implementation of assessments, and development of tools to support educators and schools. The development of the Common Core State Standards was led by the nation's governors and education commissioners through their organization--the National Governors Association Center for Best Practices (NGA) and the Council of Chief State School Officers (CCSSO). Input on the development also came from teachers, parents, school administrators, experts from around the country, and state leaders. The state and local levels are in charge of the actual implementation of the Common Core, which includes the development of curriculum materials, support materials for teachers, and the development of assessments. As of June 2014 (most recent data), 43 states have voluntarily adopted and are implementing the standards. Below is a map of the states who have adopted the CCSS and those who have not.



There are 3 key shifts in the Common Core for the mathematics standards--greater focus on fewer topics, connecting topics and thinking across grades, and the pursuit of conceptual understanding, procedural fluency, and application with equal amounts of concentration.

Great focus on fewer topics is all about deepening the information on individual topics rather than rushing to cover as many topics as possible within the curriculum. Focusing more deeply on specific topics gives students a better chance to understand a topic rather than rushing through the topics in order to complete a curriculum. This shift provides opportunities for a rich understanding of concepts, a well-developed procedural fluency, and time to apply mathematics to problems in and out of the classroom.

Connecting topics and thinking across grades is especially important in mathematics as it is a subject that is made up of interconnected concepts. The Common Core was designed so that learning is now connected across grades. Students are able to use prior knowledge from previous math classes to better build on the mathematics they are learning. This way, each of the standards is not a completely new idea, but an extension of previous learning. There is a major emphasis on connecting different mathematical topics.

The inclusion of the Standards for Mathematical Practice as part of the Common Core provides a set of characteristic that mathematically proficient students should develop while learning mathematics. These practices require teachers to be attentive to the way students learn mathematics. The pursuit of conceptual understanding, procedural fluency, and application with equal amounts of concentration require teachers to support all students in becoming increasingly proficient in mathematics. Conceptual understanding refers to learning mathematical ideas more than just facts and procedures, focusing on relationships and reasons. Procedural fluency refers to the ability to use procedures flexibly, accurately and efficiently. Application is the standards desire for students to use mathematics in situations that require mathematical reasoning while deepening conceptual understanding and procedural fluency (CCSSI, 2010).

In geometry in particular, the Common Core includes additional changes, presenting a transformational approach to congruence and similarity. While the traditional curriculum lacks any connections between middle school and high school geometry topics, the Common Core present a “seamless transition” with the use of transformations (Wu, 2012). These standards begin in eighth grade, strongly focusing on geometric transformations with emphasis on properties, sequencing, and their effect on two-dimensional figures in a coordinate plane. For example, here is the first geometry standard for eighth-grade.

> 8.G.A. Understand congruence and similarity using physical models, transparencies, or geometry software.
> 8.G.A.1 Verify experimentally the properties of rotations, reflections, and translations:
> 8.G.A.2 Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
> 8.G.A.3 Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
> 8.G.A.4: Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them (CCSS, 2010).

Much of this content is likely new to be new for many 8th grade teachers. Traditional geometry that teachers have grown up learning throughout their own mathematics classes is Euclidean based. These traditional courses have the unifying concepts of set and proof but the concepts of Euclidean geometry are not completely unifying throughout the entire course or from middle to high school. The Common Core took this into consideration and changed geometric proofs to use the mathematical characterizations of congruence, symmetry, and similarity instead. These concepts are more unifying concepts for geometry compared to the concepts of traditional geometry courses. As Wu (2012) states, the main innovation of this view of geometry "lies in nothing more than exhibiting new connections among the existing topics to clarify their mathematical relationships" (p.4). Many mathematicians and mathematics educators have come to agree with this point of view and are now advocating for transformations as the unifying topic in the geometry curriculum (Seago et al., 2013).

The definitions of congruence are very different between Euclidean geometry and transformational geometry. In Euclidean Geometry, two figures are congruent when their corresponding sides and angles are congruent, or equal to one another. In transformational geometry, two figures are congruent if you can find an isometry that transforms one shape into the other. Because these definitions of congruence are very different, the change in how to prove many geometric theorems has been drastic. Now, transformational-based proofs are different than Euclidean proofs, but transformational proofs do relate more to functions. This allows for more connections between mathematical topics, particularly in high school where the concept of functions is central to the Common Core.

The next sections present a quick introduction to transformations and in particular, to strip patterns.

Isometry

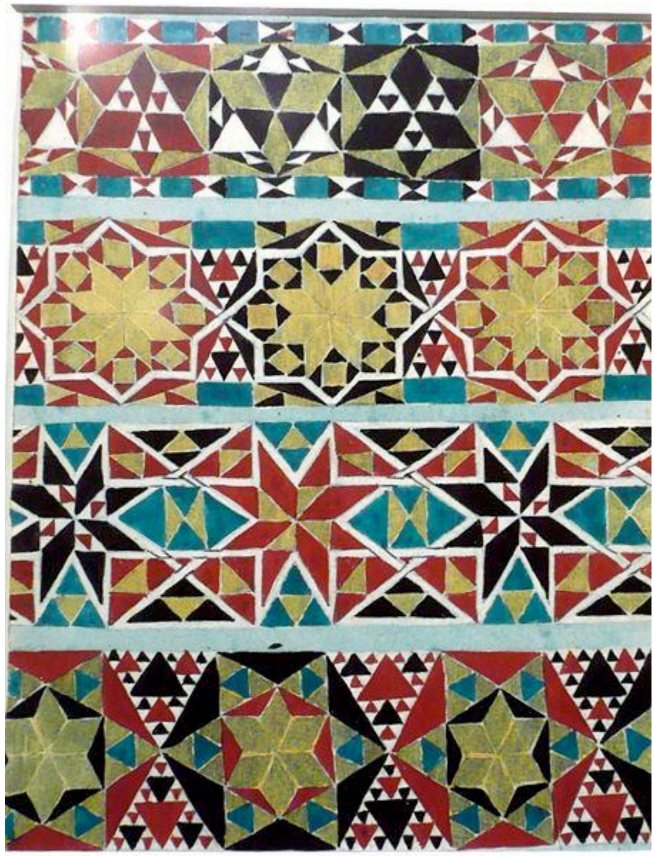
In the geometry curriculum, the idea of isometries is now widely presented. An *isometry* is a distance-preserving transformation of the real plane R^2 onto itself. Distance-preserving means that the distance between any two points in the pre-image must be the same as the distance between the images of the two points. An isometry of the plane is a linear transformation which preserves length. Another name given to an isometry is a rigid motion. Two geometric figures related by an isometry are said to be congruent. This is the definition of congruence used in the Common Core. The four types of isometries are reflections, translations, glide reflections, and rotations. Below is a list of vocabulary that will be used in the rest of the paper.

A *transformation* is a one-to-one function (also called a mapping) that maps points in the plane onto points in the plane.
A *rotation* is a transformation of the plane where a point/ figure is turned at a certain angle about a point that remains fixed.
A *reflection* is a transformation of the plane where a figure is flipped about a line, which is then considered a line of reflection. When the figure is mapped to the opposite side of the line of reflection, the perpendicular distance between any point on the figure, on one side of the line, to the line is equal to the perpendicular distance of the reflected point.

A *horizontal reflection* is a transformation of the plane where a figure is reflected over a horizontal line.
A *vertical reflection* is a transformation of the plane where a figure/point is reflected over a vertical line.
A *translation* is a transformation of the plane which shifts all points of a figure/point in a given direction through a given distance.
A *glide reflection* is a combination of the two isometries: translation and reflection. It is a transformation of the plane where a figure/point is either translated followed by being reflected or reflected followed by being translated (Krishman, 2002).
A *reflection in a point/center* is a transformation of the plane such that the point being reflected appears at an equal distance on the other side of the line that is defined by the point and the center.

M.C. Escher was a Dutch artist who incorporated mathematical depth in his work. Escher was very engaged in mathematics as his sole purpose was to understand mathematical ideas and use them in his art. He used geometry especially in creating many of his drawings and prints. His work sparked investigations by scientists and mathematics and the research Escher did in mathematics helped to later bring discoveries by mathematicians. As Escher went on with his mathematical research for his art, he eventually became obsessed with plane-filling. Escher called his tiling shapes "motifs." We can see in his work how he utilized the basic congruence-preserving transformations listed above to produce his tilings-- translations, rotations, reflections, and glide reflections. Escher discovered the tilings of many geometric figures which helped him in creating his famous artwork (Schattschneider, 2010). In a traditional mathematics curriculum in middle school, geometric transformations are taught using Escher's designs. The focus is mostly on art appreciation rather than on the mathematics behind the drawings. With the new emphasis on transformation, there is a real opportunity to understand the mathematical significance of transformations and the connections to other mathematical concepts.

In this paper, we will be looking at the tiling of strip patterns, otherwise called friezes. The tiles of strip patterns do not overlap and cover the entire strip with no gaps. The word frieze comes from architecture where is used to refer to a decorative pattern that runs horizontally, usually below a roofline. Here are some examples of strip patterns, drawn by Escher from designs of a cathedral in Ravello, Italy.



From http://euler.slu.edu/escher/index.php/Designs_from_Ravello

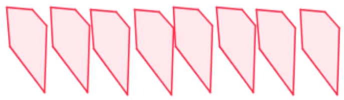
A *symmetry* of a tiling is an isometry that maps the tilings onto itself. In general, symmetry refers to a correspondence between different parts of an object. In geometry, in particular, an object is symmetric if there is a transformation that moves pieces of the object but does not change the shape. This means that objects like the strip patterns, or friezes, are symmetric. Some have reflectional, rotational, or translational symmetry, and in some cases more than one type of symmetry. The *symmetry group* of a tiling is the set that consists of all the symmetries of the tiling. The symmetry groups of strip tilings are summarized in the table below, using crystallographers' notation. There are only seven possibilities of tiling of strip patterns and there are mathematical proofs of this that will not be discussed in this paper (Krishman, 2002).

Symmetry Groups of Strip Tilings	
Four-Symbol Notation	Symmetries Present
p111	• translation
p1a1	• translation

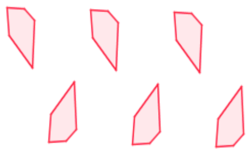
	• glide reflection
p1m1	• translation • horizontal reflection
pm11	• translation • vertical reflection
p112	• translation • rotation by 180 degrees
pma2	• translation • rotation by 180 degrees • vertical reflection • glide reflection
pmm2	• translation • rotation by 180 degrees • vertical reflection • horizontal reflection

The Common Core makes emphasis on the use of geometry software to understand congruence. Following this recommendation and to better explore these 7 symmetry groups of strip patterns, I used the software GeoGebra, which has the tools necessary to create and manipulate geometric shapes. With the “Reflect about a Line” tool, GeoGebra allows you to select a figure and a line of reflection to reflect the object over. The lines of reflection are always horizontal or vertical when working with strip patterns. With the “Rotate around a Point” tool, you can select a figure, a point to rotate about, and a degree of rotation. The angle of rotation will always be 180° when working with strip patterns. The “Translate by Vector” tool lets you select a figure to move it in the direction and magnitude of a given vector. The vectors will be always horizontal when working with strip patterns.

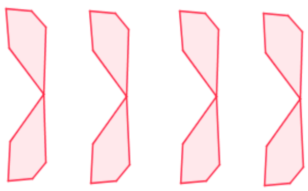
Here are examples of the seven strip patterns, created in GeoGebra. GeoGebra allows you to create dynamical objects, in this case with all seven strip patterns in one place and created with one motif. That way, when you move a point on one of the strip patterns to change the shape of the figure, every one of the 7 strip patterns changes to be that new shape as well.



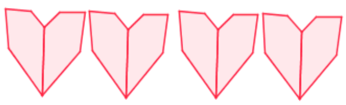
p111



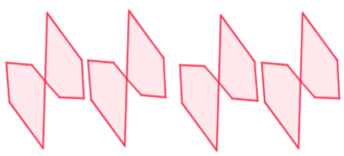
p1a1



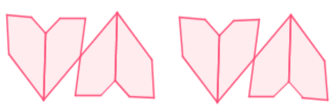
p1m1



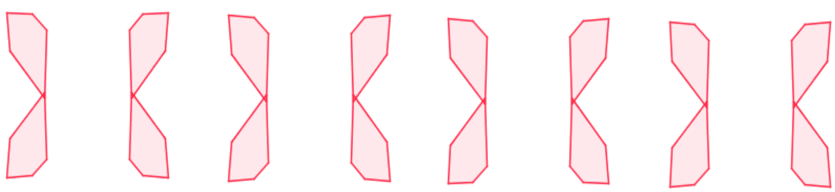
pm11



p112



pma2



pmm2

Connections between Geometry and Algebra(s)

Geometry is a unifying topic in mathematics. Sometimes it is hard for students to see this connection, as geometry and algebra are traditionally taught separately. Also in college mathematics, these courses can be disconnected from one another and sometimes the connection that exists between them is not explicitly made. Here is the connection to two areas abstract algebra and linear algebra that are not always presented.

In abstract algebra, the symmetry group of an object is composed by all the isometries under which the object is invariant. As I mentioned before, there are seven groups of symmetry for the strip patterns. Each one of these groups corresponds to a symmetry group in the abstract algebra sense, using composition of transformations as the operation.

The first group, p111, has only translations and therefore is singly generated. The abstract group corresponding is Z, the group of integers under addition. The second group, p1a1, has translations and glide reflections. In this case the abstract group is also Z. The third group, p1m1, has translations and horizontal reflections. The corresponding abstract group is ZxZ_2. The fourth group, pm11, has translations and vertical reflections. The corresponding abstract group is Dih_∞, infinite dihedral group. The fifth group, p112, has translations and 180° rotations, the abstract group is also Dih_∞. The sixth group, pma2, has translations, 180° rotations, vertical reflections and glide reflections. The abstract group is Dih_∞. Finally, the seventh group, pmm2, has translations, 180° rotations, vertical reflections and horizontal reflections. The corresponding abstract group is $Dih_\infty xZ_2$.

The geometric transformations we have studied so far are also related to other mathematical objects, matrices in linear algebra. Matrices are commonly introduced in high school algebra courses as an alternative method for solving a system of linear equations. Topics related to matrices include addition of matrices, scalar multiplication, determinants, and multiplication matrices. But each one of these operations has a geometrical representation and in the case of 2x2 matrices, some of these operations correspond to the geometric transformations mentioned before. Each point in the Cartesian plane is represented by a matrix [P] and the transformations of the point could be represented by matrices that are multiplied with the given point matrix.

Notation for doing transformations of point matrices sometimes differs between teachers and textbooks used in classrooms. In some cases, the point matrix is shown as a vertical 2x1 matrix but in others it is shown as a horizontal 1x2 matrix. Either way, the multiplication done with this matrix to the 2 x 2 transformational matrix [T] is typically the same. But depending of the point matrix notation, the order of the multiplication of the matrices might change. It may be either [P] [T] = [P'] or [T] [P] = [P'], where [P] is the point matrix being transformed, [T] is the matrix of transformation, and [P'] is the new transformed point matrix. In this paper, the point matrix referenced will be a horizontal, 1x 2 matrix and therefore we will be using [P] [T] = [P'] for the transformations. This notation means that the point matrix is being multiplied by the transformation matrix in order to find the new point matrix after the transformation has happened.

When working on the Cartesian plane, eight different transformations are of interest, these are $r_{(0,0)}$, r_x, r_y, $r_{y=x}$, $r_{y=-x}$, R_{90}, R_{180}, R_{270}. The definitions and corresponding matrices of these 8 transformations are as follows:

$r_{(0,0)}$: reflection in the origin $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

r_x: reflection in the x-axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

r_y: reflection in the y-axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$r_{y=x}$: reflection in the line y=x $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$r_{y=-x}$: reflection in the line y=-x $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

R_{90}: counter-clockwise rotation of 90° about the origin $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

R_{180}: counter-clockwise rotation of 180° about the origin $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$R_{270.}$: counter-clockwise rotation of 270° about the origin $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The elements of each one of these matrices can be found as the solution of an equation using matrices. For example, r_x: reflection in the x-axis, is the transformation that will change the coordinates of a given point P(x,y) into P'(x, -y). Using the matrix notation, we are looking for [T] that will satisfy [P] [T] = [P'], in other words,

$$[x \quad y] \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [x \quad -y]$$

From here is clear that $a = 1, b = 0, c = 0, d = -1$. Using a similar strategy, all the other matrices can be found. Using multiplication of matrices is then equivalent to composition of transformations. In this case it is important to understand the different notation used for equivalent procedures. For example, if we are doing the composition of R_{90} and $r_{y=x}$, it would be written as $R_{90°} \circ r_{y=x}$ and read as a counterclockwise rotation of 90° about the origin *following* a reflection in the line y=x. It is important to understand that the process of composition is done in reverse, where the second transformation matrix is multiplied first by the figure matrix and then the first transformation is multiplied after this. In this case, it would be [P] [$r_{y=x}$] [$R_{90°}$] where [P] is the matrix of the point being transformed.

Matrices and Geometric Transformations in the Math Classroom

In this last part of the paper, I present a lesson on the connection between geometric transformations and matrices. As part of the Common Core standards for High School, the domain of Numbers and Quantity includes a standard on performing operation on matrices and use of matrices in applications. One of the elements in the cluster reads "Work with 2 × 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area". The lesson addresses the first part of this standard. This lesson was designed for a PreCalculus BC class of 32 students. The students were mostly in 11th grade, with some in 10th grade and in 12th grade students as well.

Before this lesson, the class has been working on a unit about matrix algebra. The students learned how to multiply and add matrices. They also have learned how to take the determinant of 2x2 matrices. In previous geometry courses, the students have seen ideas of transformations. The lesson was a way to show students that not every subject in math is completely separate from one another. Students got the chance to see that math is actually interconnected in many ways, in particular how matrices and geometry are connected.

The central focus of the lesson was to understand how the eight types of geometrical transformations and its compositions relate to the corresponding matrices and its multiplications when applied to a figure, in this case a triangle, on a plane. The eight types of transformations included reflection in the origin, reflection in the x-axis, reflection in the y-axis,, reflection in line y=x, reflection in line y=-x, counterclockwise rotations of 90°, counterclockwise rotations of 180°, and counterclockwise rotations of 270°, all about the origin. During the portion of the lesson involving the composition of transformations of a triangle on a plane, students figured out how exactly a composition of transformation works and is represented using matrices.

The lesson begins by going over notation used for transformations and compositions of transformations. When doing a transformations of a point P(a,b), we use [P] = [a b] for the point matrix being transformed, [T] for the matrix of transformation, and [P'] for the new transformed point matrix.

The first activity involved transforming the point [P] = [3 1], using the 8 transformation matrices: $R_{(0,0)}$, r_x, r_y, $r_{y=x}$, $r_{y=-x}$, R_{90}, R_{180}, and R_{270}. The teacher guides the students with the first example using $r_{(0,0)}$ and the equation [P] [T] = [P']. The students' job was to find the matrix to transform P into P'. We found the matrix [T] that works for this transformation using ideas of matrix multiplication and how it works to guide us. By knowing the procedure of matrix multiplication, we looked for the transformation matrix of $r_{(0,0)}$ by thinking about what we can multiply [3 1] by in order to get the P' matrix of [-3 -1]. As a class, we were able to fill in the first transformation matrix by checking matrices that looked similar to the identity matrix to find this matrix for P'. Students would understand that by using matrix multiplication and setting up equations, we can find the matrix for this transformation. As a class, we knew that the only way we could get the newly transformed point matrix is by finding the transformation matrix that is made up of 0's, 1's and/or -1's based on the way matrix multiplication works. On the board I set up the initial outline of the equation [P] [T] = [P'] which at first we were able to fill this in with [3,1] [T] = [-3,-1]. One we had this filled in, the students were able to tell me that the transformation matrix would have to be $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in order to be able to get out the desired point matrix. And so, we were able to see that the transformation matrix for $r_{(0,0)}$ was $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ as this was the only possibility to make [3,1] $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ = [-3,-1] true.

In small groups, students explored the other seven transformations, each group in charge of one. While group work was going on, I circulated around the room, asking guiding questions to each group. Some questions I asked to groups were: "how did you find this transformation matrix?" "what were you thinking as you finding the transformation matrix?" and "what is the matrix for P' you are trying to get to from P?" After the group-work, each group presented their matrix and explained how the given matrix represents the transformation.

The second activity focused on composition of transformations, now that they found the eight transformation matrices. The students were given a copy of the coordinate plane to follow along with the work on the board to do a composition of transformations of a given triangle ΔPQR.

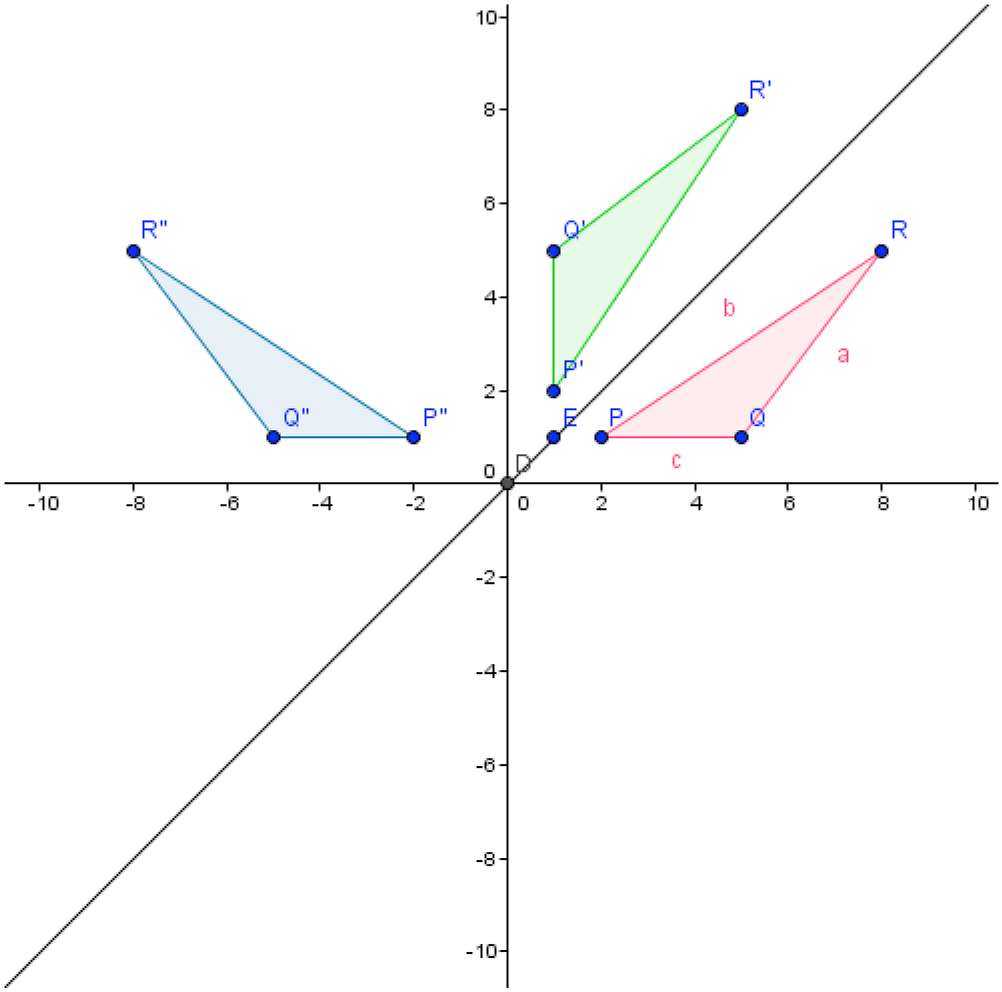
The task was to apply two transformations to the triangle. This is called "composition". The first composition for exploration is $R_{90^\circ}\ r_{y=x}$ of ΔPQR. This notation was explained as applying first $r_{y=x}$ followed by R_{90}. Another way to interpret this is to find the matrix multiplication of ΔPQR of $r_{y=x}$ first and then the multiplication with R_{90} or [ΔPQR] [$r_{y=x}$] [R_{90}], using matrix notation.

I asked the students many questions to guide the class to solve the problem together. I asked: "what are the dimensions of the matrix for [ΔPQR]?" "what is the matrix for [ΔPQR]?" "would 2x2 matrices still work for the two transformation matrices $r_{y=x}$ and R_{90}?" These questions were aimed to get students thinking about matrix dimensions in order for them to completely understand what the matrix dimensions have to be in order to be able to multiply two matrices together. Once we had the 3 matrices set up to be multiplied, I asked the students "How would you compose this?" Using the points (2,1), (5,1), and (8,5) as the coordinates for ΔPQR, we were able to together perform the two matrix multiplications necessary to find the newly transformed point matrix of the figure. When doing this, we performed the following matrix composition step by step as a class:

[ΔPQR] [$r_{y=x}$] [R_{90}] = [ΔP'Q'R']

$$\begin{bmatrix} 2 & 1 \\ 5 & 1 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 1 \\ -8 & 5 \end{bmatrix}$$

Through this portion of the lesson, the transformations of the given triangle were represented on a graph. GeoGebra was a used to present the graphs easily and to check the results done on the board. The figure below shows the GeoGebra construction. By the end of the lessons, the students were able to see how the ideas of operations with matrices can be used in transforming geometric figures on the plane. This brought the connection between algebra and geometry more clearly.



This lesson was an introduction to the world of transformational geometry using the ideas of point matrices and transformation matrices. This is a way to show how ideas from linear algebra, matrices, relate to geometric transformations. Working with matrices makes it easier to find newly transformed points and figures. This lesson presents opportunities to develop conceptual understanding, procedural fluency, and mathematical reasoning that are central to the Common Core. The conceptual understanding here refers to the connections between transformational geometry and the different representations used. The procedural fluency refers to the procedure of multiplying matrices and composing functions, as well as plotting points in the plane. Lastly, the mathematical reasoning is being able to understand the process we are following while understanding the different notations.

Conclusion

The changes in the Common Core standards with respect to the ideas of isometries in geometry brings an entire shift to the geometry curriculum. With this new perspective, it should be easier to show connections between geometry, algebra and even to functions, making the mathematics curriculum more integrated.

It is not easy to figure out exactly what is the best way to prepare teachers to follow the Common Core and but it is important to understand the shifts promoted in the Common Core. In particular in geometry, teachers need to be gaining a deeper understanding of transformations and how it connects to other topics like matrices algebra in order to fully show their students the power of mathematics.

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