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q-Series

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1 History and q-Integers

The idea of q-series has existed since at least Euler. In constructing the generating function for the partition function, he developed the basic idea of the q-exponential. From this basis, he constructed the q-logarithm, as well as numerous identities and formulae for these q-special functions. Later work in the field includes the Jacobi Theta function, generalization of orthogonal polynomials, and solutions of problems in statistical mechanics by Rogers and Ramanujan.

Of particular interest to physicists were a class of special functions called hypergeometric functions. These series, which encompass most of the familiar special functions, were fundamental to classical mechanics, however they were found to be clumsy at best when applied directly to the field of quantum mechanics. The q-analogs of these functions, however, turn out to be precisely the right tool for the job in quantum mechanics. In fact, this is largely the reason they are called q-anything; q for "quantum." Early work in the field of q-series, done by Euler and Gauss, among others, was in the area of q-hypergeometric series.

In the early 20th century, Reverend Frank Hilton Jackson worked extensively on q-series. He codified the most basic definitions of q-series, allowing easier study of basic calculus and number theory in the field. Jackson is credited with the q-analogs of numbers, derivation, and integration, and it is these topics which we will primarily consider.

Generally speaking, the q-analog of a mathematical concept is a polynomial expression in a real-valued variable q which reduces to a simple, classical object in the limit $q \to 1$. The most basic of these is the q-integer, which takes the form of the partial sum of a geometric series in q and produces a classical integer in the limit case. For $n \in \mathbb{N}$

$$[n] = \frac{q^n - 1}{q - 1} = \sum_{k=0}^{n-1} q^k \tag{1}$$

We can easily see that if $q = 1$ this sum is simply the sum of n ones, and is therefore definitively equal to n. However, it is the behavior of the general form, not the classically understood behavior of the limit, which turns out to be of interest. This definition, as well as all those which will follow, may seem arbitrary to the reader. It is the opinion of the author that, in all frankness, they are. The motivation for such definitions, almost without exception, seems to be, simply, "they work."

For a start, we might consider the behavior of the q-integers under simple arithmetic operations. Let us try adding two q-integers together. Choosing two real numbers $m < n$

$$[m] + [n] = \sum_{k=0}^{m-1} q^k + \sum_{k=0}^{n-1} q^k = 2\sum_{k=0}^{m-1} q^k + \sum_{k=m}^{n-1} q^k \tag{2}$$

Since our definition of q-integers has only unit coefficients, we immediately see that the q-integers are not closed under simple addition. In fact, there is no generically defined binary operation on the q-integers which produces the intuitive sum. In a sense, there is no single method of incrementing q-integers, since each subsequent q-integer differs from its predecessor by a different power of q. Specifically

$$[n] - [n-1] = q^{n-1} \tag{3}$$

Thus the method of adding one q-integer to another is heavily dependent on the value of the particular numbers involved. In order to obtain the type of closed addition one would expect, we would need to define a new addition operation as a function of two variables, rather than a binary operator whose structure is independent of its arguments. We could define, for example, a function Qsum as follows

$$\begin{aligned} \mathrm{Qsum}([m],[n]) &= [m] + q^m [n] \\ &= \sum_{k=0}^{m-1} q^k + q^m \sum_{k=0}^{n-1} q^k \\ &= \sum_{k=0}^{m-1} q^k + \sum_{k=m}^{m+n-1} q^k \\ &= \sum_{k=0}^{m+n-1} q^k \\ &= [m+n] \end{aligned} \tag{4}$$

Clearly the positive q-integers are closed under this new addition, and we can see that the functional form reduces to simple addition in the limit case $q \rightarrow 1$.

What about subtraction? Again we find that the q-integers are not closed under simple subtraction, so we look for another function Qdiff which might produce the expected behavior. Assuming now some real numbers $n < m$

$$\begin{aligned}
\mathrm{Qdiff}([m],[n]) &= [m] - q^{m-n}[n] \\
&= \sum_{k=0}^{m-1} q^k - q^{m-n} \sum_{k=0}^{n-1} q^k \\
&= \sum_{k=0}^{m-1} q^k - \sum_{k=m-n}^{m-1} q^k \\
&= \sum_{k=0}^{m-n-1} q^k \\
&= [m-n]
\end{aligned} \tag{5}$$

These functions may be of some interest in defining algebra using q-analogs, but the matter of negative q-integers would need to be addressed first. For the moment, we leave this topic unexplored.

Let us instead consider multiplication of the q-integers. Simply multiplying any two non-unitary q-integers together immediately provides evidence that simple multiplication is not closed either, but in this case a simple function does not immediately present itself. Luckily, the expressions produced by naive multiplication of the q-integers are sufficient to define the q-analogs of several other number-theoretical concepts.

2 Factorial & Binomial Coefficients

We define the q-factorial of an integer in the intuitive way

$$[n]! = \begin{cases} 1 & \text{if } n = 0 \\ \\ \prod_{k=1}^{n} k & \text{if } n \geq 1 \end{cases} \tag{6}$$

As the q-integers are not closed under multiplication, they certainly are not closed under the factorial function. Nonetheless, we can clearly see that the expression given by the q-factorial reduces to the classical factorial in the limit case. Let us then use this q-factorial to define an analog to another very well-known and useful tool, the binomial coefficients.

The classical binomial coefficients make us the entries in Pascal's triangle, and are given by the recursive definition

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{7}$$

As well as the explicit formula

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{8}$$

We define the q-binomial coefficients in analogy with the explicit form by

$$\begin{bmatrix} n \\ \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!} \tag{9}$$

These q-binomial coefficients deserve particular attention. Inspection of their polynomial terms immediately demonstrates that they do not constitute q-integers, yet they do resolve to classical integers in the limit case, just as the q-factorials before them. So it seems that we have defined several classes of q-analogs to the concept of an integer.

We can think of these different classes as being related to the issue of partitioning the integers. Since the limit as $q \to 1$ of any power of q is itself 1, we can view polynomials in q whose coefficients sum to n as a partition of n. In other words, any polynomial in q is equal to the sum n of its coefficients in the limit case, and can therefore be considered a q-analog of n. However, we will continue with the definitions we have given, referring exclusively to $[n]$ as q-integers, etc.

If we wish to reconstruct the nature of binomial coefficients in this new context, we must find a recursive definition. We are able to accomplish this task by means of the Qsum function, implemented where simple addition would be used in the proof of the classical recursive definition.

$$\begin{aligned}
\frac{[n]!}{[n-k]![k]!} &= \frac{[n-1]![n]}{[n-k]![k]!} \\
&= \frac{[n-1]!\mathrm{Qsum}([k],[n-k])}{[n-k]![k]!} \\
&= \frac{[n-1]!([k]+q^k[n-k])}{[n-k]![k]!} \\
&= \frac{[n-1]!}{[n-k]![k-1]!} + q^k \frac{[n-1]!}{[n-k-1]![k]!} \\
&= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}
\end{aligned} \tag{10}$$

Similarly,

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} \tag{11}$$

We refer to these formulae as the q-Pascal identities.

3 Fibonacci and Triangular Numbers

With the q-binomial coefficients defined, we can consider some other number theoretical q-analogs: The q-Fibonacci numbers and the q-triangular numbers.

The q-Fibonacci numbers are defined by a recurrence relation

$$\begin{aligned} \tilde{F}_0 &= 0 \\ \tilde{F}_1 &= 1 \\ \tilde{F}_n &= \tilde{F}_{n-1} + q^{n-1}\tilde{F}_{n-2} \end{aligned} \tag{12}$$

The classical Fibonacci numbers can be obtained from Pascal's triangle, and therefore from the binomial coefficients. We can obtain the $n+1$st Fibonacci number F_{n+1} by summing a diagonal of the Pascal's triangle as follows

$$F_{n+1} = \sum_{0 \leq 2k \leq n} \binom{n-k}{k} \tag{13}$$

We have previously defined the q-binomial coefficients, so it is reasonable to expect that a similar summation identity should exist relating them to the q-Fibonacci numbers. We will prove the existence of such an identity by induction. We begin with the statement of our new identity as the induction hypothesis

$$\tilde{F}_{n+1} = \sum_{0 \le 2k \le n} q^{k^2+k} \begin{bmatrix} n-k \\ k \end{bmatrix} \tag{14}$$

The value of $\tilde{F}_0$ is taken as definition, but $\tilde{F}_1$ can easily be shown to follow this definition

$$\tilde{F}_1 = q^0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \tag{15}$$

Let us assume, then, that our hypothesis is true for all indices less than some N. Then we can use our recurrence relation in conjunction with the induction hypothesis as follows

$$\begin{aligned} \tilde{F}_N &= \tilde{F}_{N-1} + q^{N-1}\tilde{F}_{N-2} \\ &= \sum_{0 \le 2k \le N-2} q^{k^2+k} \begin{bmatrix} N-k-2 \\ k \end{bmatrix} + q^{N-1} \sum_{0 \le 2k \le N-3} q^{k^2+k} \begin{bmatrix} N-k-3 \\ k \end{bmatrix} \\ &= \sum_{0 \le 2k \le N-2} \left(q^{k^2+k} \begin{bmatrix} N-k-2 \\ k \end{bmatrix} + q^{k^2-k+N-1} \begin{bmatrix} N-k-2 \\ k-1 \end{bmatrix} \right) \\ &= \sum_{0 \le 2k \le N-2} q^{k^2+k} \left(\begin{bmatrix} N-k-2 \\ k \end{bmatrix} + q^{N-2k-1} \begin{bmatrix} N-k-2 \\ k-1 \end{bmatrix} \right) \end{aligned} \tag{16}$$

By the q-Pascal identities, we can rewrite the term in parentheses as a single q-binomial coefficient

$$\begin{bmatrix} N-k-1 \\ k \end{bmatrix} = \begin{bmatrix} N-k-2 \\ k \end{bmatrix} + q^{(N-k-1)-(k-1)} \begin{bmatrix} N-k-2 \\ k-1 \end{bmatrix} \tag{17}$$

Thus, with some reindexing, we obtain exactly the relation intended for $\tilde{F}_N$

$$\tilde{F}_N = \sum_{0 \leq 2k \leq N} q^{k^2+k} \begin{bmatrix} N-k-1 \\ k \end{bmatrix} \tag{18}$$

Another series which arises from diagonal sums of Pascal's triangle is the triangular numbers. This series, denoted t_n, is defined as the sum of the first n natural numbers

$$t_n = \sum_{k=0}^{n} \binom{k}{1} = \sum_{k=0}^{n} k \tag{19}$$

The closed form of the triangular numbers is given by

$$t_n = \frac{n(n+1)}{2} \tag{20}$$

We define the q-triangular numbers by a similar set of equations

$$\tilde{t}_n = \sum_{k=0}^{n} q^k [k] = \frac{q[k][k+1]}{[2]} \tag{21}$$

To see that this relation holds, we simplify and evaluate the summation directly.

$$\begin{aligned} \sum_{k=0}^{n} q^k [k] &= \sum_{k=0}^{n} q^k \frac{q^k - 1}{q-1} \\ &= \frac{1}{q-1} \left(\sum_{k=0}^{n} q^{2k} - \sum_{k=0}^{n} q^k \right) \end{aligned} \tag{22}$$

We notice that the first sum in the parentheses is the partial sum of a geometric series with ratio q^2, while the second is exactly the definition of $[n+1]$. We then factor the partial sum as the difference of two squares and begin to simplify all terms to q-integers

$$\begin{aligned}
\tilde{t}_n &= \frac{1}{q-1}\left(\frac{1-q^{2n+2}}{1-q^2} - [n+1]\right) \\
&= \frac{1}{q-1}\left(\frac{1-q^{n+1}}{1-q}\frac{1+q^{n+1}}{1+q} - [n+1]\right) \\
&= \frac{[n+1]}{q-1}\left(\frac{1+q^{n+1}}{1+q} - 1\right) \\
&= \frac{[n+1]}{q-1}\frac{q^{n+1}-q}{1+q} \\
&= \frac{[n+1]}{[2]}\frac{q(q^n-1)}{q-1} \\
&= q\frac{[n][n+1]}{[2]}
\end{aligned} \tag{23}$$

The Fibonacci and triangular numbers are just two examples of the myriad series which can be obtained by inspection and manipulation of Pascal's triangle. The triangular numbers in particular are the simplest case of such a family of series known as the figurate numbers. With the definition of the q-binomial coefficients, it is possible, though not nearly as aesthetically pleasing, to construct a triangular array whose entries are the polynomial q-binomial coefficients.

By inspecting such an array, it may be possible to find further q-analogs and identities, but the task is daunting. In the case of the classical Pascal's triangle, the integer entries are easy to interpret, and patterns quickly become obvious. In the q-analog, there is no such intuition to be had. Finding patterns in this case would be a matter of extensive inspection, almost definitely with the aid of a computer algebra system such as Mathematica. This study is left to the reader.

4 Derivative

The q-analog of the derivative, historically called the Jackson Derivative D_q, is given by the almost-familiar formula

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \tag{24}$$

This operator yields the classical derivative in the limit case, as each of our other q-analogs have done. In fact, the usual definition of the derivative is given by a distinct, ultimately equivalent limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{df}{dx} = \lim_{q \to 1} \frac{f(qx) - f(x)}{(q-1)x} \tag{25}$$

D_q has the following convenient property:

$$D_q x^n = [n] x^{n-1} \tag{26}$$

Essentially, D_q treats q-integers the same way the classical derivative treats classical integers. Although D_q is linear and has many of the same properties as the classical derivative, it lacks a generalized chain rule. The expected relation

$$D_q f(g(x)) = (D_q f)(g(x)) \cdot D_q g(x) \tag{27}$$

Only generally holds for the simple monomial functions given in Eq. 14. As such, it is necessary to use polynomial expansions of more complicated functions in order to define their q-derivatives. To this end, we must define the q-Taylor Series.

5 Taylor Series

The familiar Taylor Series expansion of a function is a specific case of a more general rule. As proven in [**?**], any linear operator which reduces the order of polynomial input can be used to expand a function around a given point a. For example, given the D_q operator and some function f of order N, we can find polynomials $P_k(x)$ such that

$$f(x) = \sum_{k=0}^{N} (D_q^n f)(a) P_k(x) \tag{28}$$

in a neighborhood of a.

In order to determine the type of polynomials we must use, we have a set of three rules.

$$P_k(a) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases} \tag{29}$$

$$\mathbf{deg} P_k = k \tag{30}$$

$$D_q P_k(x) = P_{k-1} \tag{31}$$

Since we know already that D_q reduces the order of polynomial input, the third requirement is simply a restriction on the coefficients of the polynomials chosen. In order to find the family of polynomials which satisfy these conditions, we must return to the q-analog for the binomial expansion.

Since we have q-binomial *coefficients*, let's look at a q-analog for binomial expansion

$$(x-a)_q^n = \begin{cases} 1 & \text{if } n = 0 \\ (x-a)(x-qa)...(x-q^{n-1}a) & \text{if } n \geq 1 \end{cases} \tag{32}$$

It should be easy to see that this resolves to classical binomial expansion in the limit case. An important property of these q-binomials is given by

$$D_q(x-a)_q^n = [n](x-a)_q^{n-1} \tag{33}$$

Now consider a family of polynomials P_k defined by

$$P_k(x) = \frac{(x-a)_q^k}{[k]!} \tag{34}$$

If we apply the Jackson derivative to these polynomials, we see

$$\begin{aligned} D_q P_k(x) &= D_q \frac{(x-a)_q^k}{[k]!} \\ &= \frac{[k](x-a)_q^{k-1}}{[k]!} \\ &= \frac{(x-a)_q^{k-1}}{[k-1]!} \\ &= P_{k-1}(x) \end{aligned} \tag{35}$$

Thus our P_k satisfy conditions (29)-(31), and are therefore candidates for the q-Taylor expansion. We can now write the q-Taylor expansion of a function f about a point a:

$$f(x) = \sum_{k=0}^{N} (D_q^k f)(a) \frac{(x-a)_q^k}{[k]!} \tag{36}$$

6 Trigonometric & Exponential functions

Special functions like the exponential and trigonometric functions have well-known Taylor series expansions. With the definition of the q-Taylor series, we can define new functions with analogous expansions whose behavior will be similar to the classical functions. For example

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{37}$$

Since we have the q-factorial, it is a reasonable jump to suggest that the q-exponential could be represented by

$$e_q^x = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} \tag{38}$$

Yet again, this definition resolves to the original in the limit case. However, in order for this to be considered a useful definition, we wish to know whether its behavior under the Jackson derivative is similar to the classical exponential's behavior under traditional differentiation.

$$\begin{aligned} D_q e_q^x &= \sum_{k=0}^{\infty} \frac{D_q x^k}{[k]!} \\ &= \sum_{k=1}^{\infty} \frac{[k] x^{k-1}}{[k]!} \\ &= \sum_{k=1}^{\infty} \frac{x^{k-1}}{[k-1]!} \\ &= \sum_{k=0}^{\infty} \frac{x^k}{[k]!} \\ &= e_q^x \end{aligned} \tag{39}$$

This is precisely the behavior we would expect from the exponential function, so we can consider this a valid definition of a q-exponential function. This is not the only such definition, and in fact another q-exponential definition turns out to be of equal use.

$$E_q^x = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} q^{\frac{k(k-1)}{2}} \tag{40}$$

This new function behave slightly differently under D_q

$$D_q E_q^x = E_q^{qx} \tag{41}$$

But we note that this difference vanishes in the limit $q \to 1$.

Now, again by analogy with classical definitions, we find the q-sine and q-cosine functions. We have the following definitions for sine and cosine in terms of the classical exponential:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{42}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{43}$$

So, using our two q-exponential functions, we can define two distinct pairs of q-trigonometric functions

$$\sin_q(x) = \frac{e_q^{ix} - e_q^{-ix}}{2i} \tag{44}$$

$$\cos_q(x) = \frac{e_q^{ix} + e_q^{-ix}}{2} \tag{45}$$

And

$$\mathrm{Sin}_q(x) = \frac{E_q^{ix} - E_q^{-ix}}{2i} \tag{46}$$

$$\mathrm{Cos}_q(x) = \frac{E_q^{ix} + E_q^{-ix}}{2} \tag{47}$$

These functions behave as expected under D_q. In particular

$$D_q \sin_q(x) = \cos_q(x) \tag{48}$$

$$D_q \cos_q(x) = -\sin_q(x) \tag{49}$$

And

$$D_q \mathrm{Sin}_q(x) = \mathrm{Cos}_q(qx) \tag{50}$$

$$D_q\text{Cos}_q(x) = -\text{Sin}_q(qx) \tag{51}$$

Where, as before, the extra q picked up by the Sin_q and Cos_q functions vanishes in the limit. We also note that, after a bit of algebra,

$$\sin_q(x)\text{Sin}_q(x) + \cos_q(x)\text{Cos}_q(x) = 1 \tag{52}$$

This gives a q-analog for the classical identity $\sin^2(x) + \cos^2(x) = 1$.

7 The Jackson Integral

Jackson defined a method of q-integration as the inverse of the D_q operator. The Jackson integral, like the Jackson derivative, does not involve a limit in its definition. As a result, q-integration is a discrete summation, rather than the continuous type of integral with which we are familiar from classical calculus.

The definite integral is defined as follows:

$$\int_0^a f(x)d_qx = a(1-q)\sum_{k=0}^{\infty} q^k f(q^k a) \tag{53}$$

Which can be seen as a Riemann-like sum over a varying interval width. Here d_q is understood to be the small difference in input roughly equivalent to $(q-1)x$. This equation quickly leads to a rule for integrating from one non-zero input to another, which subsequently allows us to define a q-antiderivative of $f(x)$.

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx \tag{54}$$

Which, by analogy with the Fundamental Theorem of Calculus, leads to the conclusion

$$\int f(x)d_qx = x(1-q)\sum_{k=0}^{\infty} q^k f(q^k x) \tag{55}$$

Let us test this method of integration with an elementary q-function. If we are to accept the Jackson integral, we require that it precisely reverse the action of the Jackson derivative up to multiplication by q. We will evaluate the antiderivative of x^n by way of an example.

$$\begin{aligned}
\int x^n d_qx &= x(1-q)\sum_{k=0}^{\infty} q^k (q^k x)^n \\
&= x^{n+1}(1-q)\sum_{k=0}^{\infty} q^{(n+1)k} \\
&= x^{n+1}\frac{1-q}{1-q^{n+1}} \\
&= \frac{x^{n+1}}{[n+1]}
\end{aligned} \tag{56}$$

We can see that the Jackson derivative, integral, and our ensemble of q-analogs behave altogether like classical calculus. This parallel behavior positions the q-calculus as a broadly applicable generalization of classical calculus. The tools and identities formulated in classical calculus and their uses in classical mechanics are mirrored in q-calculus and do in fact find use in quantum mechanics.