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Cornered Circles

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Introduction:

The circle, the set of points equal distance from a given point, is a round, perfectly symmetrical object in two dimensional Euclidian Space. What if we redefined distance, the metric, in such a way that circles were not circles, but rather cornered circles? What would happen to the circumference? Of course, because circumference is dependent on measuring distance the change in metric would also affect the circumference measurement. Given a symmetric convex set, it is possible to define a metric space that has the boundary of the set as a unit circle. The main result of this paper is to find the circumference of every even edged regular polygon unit circle in the corresponding metric space.

Section 1: Metric and Norm

In this section we explore the basics of metrics and norms to better understand how they relate to each other. We will prove that a metric can be obtained from any norm, a fact we'll want to use a great deal as we move forward.

Definition 1.1: Metric.

A metric on $\mathbb{R}^2$ is a function $d_x : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ that measures the distance between two points in $\mathbb{R}^2$. The function d_x must satisfy the following properties for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^2$.:

1) $d_x[\boldsymbol{a}, \boldsymbol{b}] \geq 0$
2) $d_x[\boldsymbol{a}, \boldsymbol{b}] = 0$ iff $\boldsymbol{a} = \boldsymbol{b}$
3) $d_x[\boldsymbol{a}, \boldsymbol{b}] = d_x[\boldsymbol{b}, \boldsymbol{a}]$
4) $d_x[\boldsymbol{a}, \boldsymbol{b}] + d_x[\boldsymbol{b}, \boldsymbol{c}] \geq d_x[\boldsymbol{a}, \boldsymbol{c}]$

A *metric space* is the pair (M, d_x) where M is a set and d_x is a metric on set M. In the metric definition M was $\mathbb{R}^2$.

A metric fundamentally tells us the distance from any two objects in the set. Distances are always seen in absolute terms, never negative. This is property one. Property two of metrics makes sure the distance from any two points is always positive unless you are measuring from a point to itself, in which case it's always 0. Property three says that going from object A to object B is the same distance in reverse. Property 4 is the triangle inequality; going through another point between A and C will never shorten the distance. When most people talk about a metric they often mean the Euclidian metric, so we'll define that here.

Definition 1.2: Euclidian Metric.

The Euclidean metric on $\mathbb{R}^2$ is given by $d_E[(a,b),(c,d)] = \sqrt{(a-c)^2 + (b-d)^2}$ where $a, b, c, d \in \mathbb{R}$. The Euclidian metric space is the pair $E = (\mathbb{R}^2, d_E)$.

Metrics in general don't have to be as well behaved as the Euclidian metric, meaning that they can do some unexpected things. For starters metrics don't require any kind of completeness axiom on the set they are applied to. This allows for metrics to exists on discrete and finite sets. Another thing that can go wrong is that metrics aren't always invariant under translations. For example if we consider the real number line with a nonstandard metric, the distance from 3 to 4 might not be the same as the distance from 4 to 5. The distance defining object that does have all the extra properties you would expect a metric to have is a norm.

Definition 1.3: Norm.

A norm is a function $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ that measures the distance from the origin to any point in $\mathbb{R}^2$ such that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$ and $c \in \mathbb{R}$:

1) $\| \boldsymbol{a} \| \geq 0$
2) $\| \boldsymbol{a} \| = 0$ iff $\boldsymbol{a} = (0, 0)$
3) $\| c\boldsymbol{a} \| = |c| \| \boldsymbol{a} \|$
4) $\| \boldsymbol{a} \| + \| \boldsymbol{b} \| \geq \| \boldsymbol{a} + \boldsymbol{b} \|$

A *normed linear space* is the ordered pair $(N, \| \cdot \|)$ where *N* is a vector space and $\| \cdot \|$ is a norm on set *N*.

Since we are claiming that norms do more than metrics do, we should be able to create metrics from norms. We will prove just that.

Proposition 1.4: Metric from a Norm.

Given any norm $\| \cdot \|_x$ we can define a metric d_x on $\mathbb{R}^2$ by $d_x[\boldsymbol{p}, \boldsymbol{q}] = \| \boldsymbol{q} - \boldsymbol{p} \|_x$.

Proof: Obviously norm (1) and (2) implies metric (1) and (2) respectively. Norm (3) gives us metric (3) by

$$d_x[\boldsymbol{p}, \boldsymbol{q}]$$

$$= \| \boldsymbol{q} - \boldsymbol{p} \|_x$$

$$= \|(-1)(\boldsymbol{p} - \boldsymbol{q}) \|_x$$

$$= |-1| \| \boldsymbol{p} - \boldsymbol{q} \|_x$$

$$= \| \boldsymbol{p} - \boldsymbol{q} \|_x$$

$$= d_x[\boldsymbol{q}, \boldsymbol{p}].$$

Metric (4) comes from norm (4) by

$$d_x[\boldsymbol{a},\boldsymbol{b}] + d_x[\boldsymbol{b},\boldsymbol{c}]$$

$$= \| \boldsymbol{b} - \boldsymbol{a} \| + \| \boldsymbol{c} - \boldsymbol{b} \|$$

$$\geq \| (\boldsymbol{b} - \boldsymbol{a}) + (\boldsymbol{c} - \boldsymbol{b}) \|$$

$$= \| \boldsymbol{c} - \boldsymbol{a} \|$$

$$= d_x[\boldsymbol{a},\boldsymbol{c}].$$

Since a norm presumes more structure than a metric does, a metric which comes from a norm will have some extra properties. One such property is Proposition 1.5, which shows that norms are translation invariant.

Proposition 1.5: Given a metric from a norm, $\| \cdot \|_x$, $d_x[\boldsymbol{a},\boldsymbol{b}] = d_x[\boldsymbol{a}+\boldsymbol{c},\boldsymbol{b}+\boldsymbol{c}]$ for $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in \mathbb{R}^2$

Proof: From our definitions:

$$d_x[\boldsymbol{a},\boldsymbol{b}] = \| \boldsymbol{a} - \boldsymbol{b} \|_x = \| \boldsymbol{a} + \boldsymbol{c} - \boldsymbol{b} - \boldsymbol{c} \|_x = \|(\boldsymbol{a}+\boldsymbol{c}) - (\boldsymbol{b}+\boldsymbol{c})\|_x = d_x[\boldsymbol{a}+\boldsymbol{c},\boldsymbol{b}+\boldsymbol{c}].$$

As an add-on to Proposition 1.5 we have Proposition 1.6.

Proposition: 1.6: If a metric satisfies $d_x[\boldsymbol{a},\boldsymbol{b}] = d_x[\boldsymbol{a}+\boldsymbol{c},\boldsymbol{b}+\boldsymbol{c}]$ for $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in \mathbb{R}^2$ then $d_n[\boldsymbol{a},\boldsymbol{b}] = d_x[-\boldsymbol{a},-\boldsymbol{b}]$.

Proof: $d_x[\boldsymbol{a},\boldsymbol{b}] = d_x[\boldsymbol{b},\boldsymbol{a}] = d_x[\boldsymbol{0},\boldsymbol{a}-\boldsymbol{b}] = d_x[-\boldsymbol{a},-\boldsymbol{b}]$.

At this point we're writing these proofs to show we can. It's good to know how to use the properties available to find new ones as this is our only way to progress to something new. The progression we're going to move towards involves circles. This should be thought of as the set of points distance r from a given point. Since we plan to explore other norms which give different ways to measure distance we shouldn't expect our circles to always be round. In fact, we'll be considering cornered circles.

Definition 1.7: Ball and Circle.

Let (X, d_x) be a metric space. The *ball* with center *x* and radius *r* is defined by

$$B(x,r) = \{y \in X \mid d_x(x,y) \leq r\}.$$

The *circle* with center *x* and radius *r* is defined by $C(x,r) = \{y \in X \mid d_x(x,y) = r\}$.

Section 2: Metric Isometries and Norm Preserving Maps

In this section we'll define and explore the basics of isometries. We'll also introduce the taxi cab norm and max norm. Then we will prove there exists an isometry between these two norms to serve as an example. We will also show that a norm preserving map implies an isometry exists, and that $Isom(X)$ is a group.

Definition 2.1: Isometry.

Let *X* and *Y* be metric spaces, then the map $f : X \to Y$ is an isometry if for any $a, b \in X$:

$$d_Y[f(a), f(b)] = d_X[a, b].$$

We say two metrics are isometric when an isometry exists between them. This in essence means that the map is distance preserving. If two points are distance x apart, then their images will be distance x apart. The isometry is one of the most important tools of this paper to prove the results we want. We will think of isometries in two ways, a map from one metric to another, or a map from a metric onto itself. A non-trivial isometry onto itself implies there is some kind of symmetry of the metric space.

Lemma 2.2: An isometry is one-to-one.

Proof: Let X and Y be metric spaces, and $f : X \to Y$ an isometry. Let $a, b \in X$ so that $f(a) = f(b)$. If $f(a) = f(b)$, then $d_Y[f(a), f(b)] = d_Y[f(a), f(a)] = 0$ by property 2 of metrics. Then since f is an isometry, we know $d_X[a, b] = 0$ as well. Again by property 2 of metrics, $a = b$. Therefore f is one-to-one.

When working with isometries we'll want our mappings to be bijections (one-to-one and onto) so we can invert the isometry. It turns out that onto isn't guaranteed from the definition. Fortunately, we'll only ever consider linear functions which are onto. Whenever we talk about isometries we're going to assume that the function is onto. We don't need to assume that isometries are one-to-one as well since that's a consequence of their definition.

Definition 2.3: $Isom(X)$.

For any metric space X, $Isom(X) = \{f: X \to X \mid f \text{ is an onto isometry}\}$.

Notice Definition 2.3 requires that the isometry is onto and only considers isometries onto itself. Since these types of isometries imply some sort of symmetry of the metric space it should be no surprise that abstract algebra shows up in the following theorem.

Theorem 2.4: $Isom(X)$ is a group under function composition.

Proof: For closure, let $f(x)$, $g(x) \in Isom(X)$; x, $y \in X$ and (X, d_x) is a metric space. We show that $f(g(x)) \in Isom(X)$, rather, $d_x[x,y] = d_x[f\big(g(x)\big), f\big(g(y)\big)]$. The distance $d_x\big[f\big(g(x)\big), f\big(g(y)\big)\big] = d_x[g(x), g(y)]$ since $f(x) \in Isom(X)$, and $d_x[g(x), g(y)] = d_x[x,y]$ since $g(x) \in Isom(X)$. Thus $Isom(X)$ is closed.

Function composition by nature is associative, and so $Isom(X)$ is associative.

To show $Isom(X)$ has an identity element, we will verify that $f(x) = x$ is the identity element for $Isom(X)$. To show $f(x) \in Isom(X)$, note that $d_x[f(x), f(y)] = d_x[x,y]$ since $f(x) = x$, so $f(x) \in Isom(X)$. Also $f(g(x)) = g(x) = g(f(x))$ for any $g(x) \in Isom(X)$. Thus f(x) is the identity element and so $Isom(X)$ has an identity element.

This also shows that $Isom(X)$ is nonempty.

Now we show that every element of $Isom(X)$ has an inverse element. Let $f(x) = x$ be the identity element and $g(x) \in Isom(X)$. Since $g(x) \in Isom(X)$, we know that $g(x)$ is one-to-one by Lemma 2.2 and onto by Definition 2.3. This implies that there exists a $g^{-1}(x)$ that is the inverse of $g(x)$ so that $g^{-1}(g(x)) = f(x) = g(g^{-1}(x))$. We just have to show $g^{-1}(x) \in Isom(X)$, rather $d_x[g^{-1}(x), g^{-1}(y)] = d_x[x,y]$. Since $g(x) \in Isom(X)$, $d_x[g^{-1}(x), g^{-1}(y)] = d_x[g(g^{-1}(x)), g(g^{-1}(y))]$. Also, $d_x[g(g^{-1}(x)), g(g^{-1}(y))] = d_x[x,y]$ because $g\big(g^{-1}(x)\big) = f(x) = x$, Thus all elements of $Isom(X)$ have an inverse element of $Isom(X)$.

Therefore $Isom(x)$ is a group under function composition.

Now we will change gears and turn our focus from isometries that map metrics onto themselves to isometries that map between two different metric spaces. This idea is more complex in nature, so we will use examples to help us understand just what an isometry is doing. To do an example we'll need specific examples of metric spaces.

Definition 2.5: Taxi Cab Norm.

The normed linear space notated by $(\mathbb{R}^2, \|\cdot\|_t)$ where $\|\cdot\|_t = Sum\{|x|, |y|\}$ for $(x, y) \in \mathbb{R}^2$ are the taxi cab normed linear space and taxi cab norm respectively.

This one is called the taxi cab norm because city blocks keep taxi cab drivers from making any diagonal movement. This means if a taxi needs to get x block east and y blocks north his total distance traveled is the sum of the absolute values of x and y.

Definition 2.6: Max Norm.

The normed linear space notated by $(\mathbb{R}^2, \|\cdot\|_m)$ where $\|\cdot\|_m = Max\{|x|, |y|\}$ for $(x, y) \in \mathbb{R}^2$, are the max normed linear space and max norm respectively.

We said we wanted to give an example of an isometry between metric spaces but we just defined a pair of normed linear spaces. From Proposition 1.4 we can make the taxi cab metric and max metric. We won't bother writing these metric spaces down. Isometries on metrics take a lot of work since there are four free variables involved when on $\mathbb{R}^2$. We can find an easier way to show the isometry exists.

Definition 2.7: Norm Preserving Map.

Given normed spaces $(N, \|\cdot\|_N)$ and $(M, \|\cdot\|_M)$, $f: N \to M$ is a norm preserving map if for all $n \in N$, $\| f(n) \|_M = \| n \|_N$.

This is our way out of working with isometries. It will clearly be much easier than an isometry since when N and M are $\mathbb{R}^2$ we will only need to work with two free variables. First we need to confirm that a norm preserving map is also an isometry.

Lemma 2.8: If $f : (\mathbb{R}^2, \| \cdot \|_1) \to (\mathbb{R}^2, \| \cdot \|_2)$ is a linear, norm preserving map, then f is an isometry between $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_2)$.

Proof: We will show $d_1[f(a), f(b)] = d_2[a, b]$.

$$d_1[f(a), f(b)] = \| f(b) - f(a) \|_1 = \| f(b - a) \|_1 = \| b - a \|_2 = d_2[a, b].$$

The second equality follows as f is linear.

We can easily see why such a lemma would be true since the metrics are defined by the norms. This also means we can only use this lemma when the metrics are defined by the norms. For this paper that will always be true. We also had to assume that the map was linear, which will also always be the case.

Theorem 2.9: Taxi Cab Metric and Max Metric are isometric.

Proof: Let $X = (\mathbb{R}^2, \| \cdot \|_m)$ be the max normed space and $Y = (\mathbb{R}^2, \| \cdot \|_t)$ be the taxi cab normed space. Let $a, b \in \mathbb{R}$. Let $f: X \to Y$ be the map defined by $f(a, b) = \left(\frac{1}{2}a - \frac{1}{2}b, \frac{1}{2}a + \frac{1}{2}b\right)$. We claim that f is an norm-preserving map, $\| f(a, b) \|_t = \| (a, b) \|_m$.

$$\| f(a, b)\|_t = \left\| \left(\frac{1}{2}a - \frac{1}{2}b, \frac{1}{2}a + \frac{1}{2}b\right) \right\|_t$$

$$= Sum\left\{\left|\frac{1}{2}a - \frac{1}{2}b\right|, \left|\frac{1}{2}a + \frac{1}{2}b\right|\right\}$$

$$= \frac{1}{2}|a - b| + \frac{1}{2}|a + b|.$$

If $|a| \geq |b|$ and $a \geq 0$, then

$$\| f(a,b) \|_t = \frac{1}{2}|a-b| + \frac{1}{2}|a+b|$$

$$= \frac{1}{2}(a-b) + \frac{1}{2}(a+b)$$

$$= a$$

$$= Max\{|a|, |b|\}.$$

If $|a| \geq |b|$ and $a < 0$, then

$$\| f(a,b) \|_t = \frac{1}{2}|a-b| + \frac{1}{2}|a+b|$$

$$= -\frac{1}{2}(a-b) - \frac{1}{2}(a+b)$$

$$= -a$$

$$= Max\{|a|, |b|\}.$$

If $|a| < |b|$ and $b \geq 0$, then

$$\| f(a,b) \|_t = \frac{1}{2}|a-b| + \frac{1}{2}|a+b|$$

$$= -\frac{1}{2}(a-b) + \frac{1}{2}(a+b)$$

$$= b$$

$$= Max\{|a|, |b|\}.$$

If $|a| < |b|$ and $b < 0$, then

$$\| f(a,b) \|_t = \frac{1}{2}|a-b| + \frac{1}{2}|a+b|$$

$$= \frac{1}{2}(a-b) - \frac{1}{2}(a+b)$$

$$= -b$$

$$= Max\{|a|, |b|\}.$$

This covers all cases for choosing a and b so we have sown that $\| f(a,b) \|_t = \| (a,b) \|_m$ since $\| (a,b) \|_m = Max\{|a|, |b|\}$. Thus f is a norm-preserving map and thus an isometry by Lemma 2.8. Therefore the Taxi Cab Metric and the Max Metric are isometric.

Now that we've found an isometry, what more do we know about these two metrics? We essentially learned that most of the *stuff* about these two metrics are going to either be similar or exactly the same. What stuff exactly is hard to say but can be examined on a case by case basis. For example we can look at circles in both of these metrics. Below is an image of the taxi cab, max, and Euclidian circles, with radius one, centered at the origin in red, blue, and black respectively.

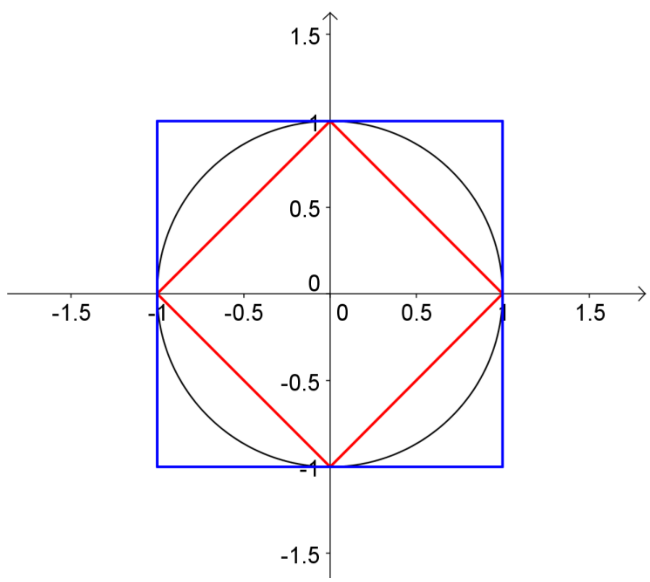

Figure 1

As promised we have cornered circles. We can see that the circles for both the taxi cab and max metric aren't just both cornered, but they are both squares. Of course we'll investigate this relationship further.

Section 3: Convex Sets, Curves and the Hierarchy

In section 3 we will show that there is even something more elementary to the norm, the convex set. We will reason a convex set is given by its boundary. Then from there we will establish the hierarchy, an ordering of spaces and distance preservation. We will also define the unit circle intersection and hint at the extreme usefulness it holds.

Definition 3.1: Unit Ball.

Let $(\mathbb{R}^2, d_x)$ be a metric space. The *unit ball* with center $\mathbf{0}$ and radius 1 is

$$B(\mathbf{0}, 1) = \{y \in \mathbb{R}^2 |\ d_x(\mathbf{0}, y) \leq 1\}.$$

If this metric was defined by a norm, then

$$B(\mathbf{0}, 1) = \{y \in \mathbb{R}^2 |\ \| \boldsymbol{y} \|_x \leq 1\}.$$

This simple concept could have been defined much earlier. It's defined now because it has relevance to the following definitions. The unit ball is the set of point less than or equal 1 distance away from the origin.

Definition 3.2: Convex sets define norms.

Given a set K and $\boldsymbol{q} \in \mathbb{R}^2$ with the following properties:

1) K is a closed, bounded set with $\mathbf{0}$ in the interior,

2) K is symmetric with respect to $\mathbf{0}$,

3) K is convex,

Then the function $\| \boldsymbol{q} \|_K \coloneqq \inf\{t > 0 \mid \boldsymbol{q}/t \in K\}$ is a norm on $\mathbb{R}^2$ with K as the unit ball. Unless otherwise specified a *convex set* will have properties (1) and (2) as well as being convex.

We will not prove this definition and instead reference [1]. Essentially the infimum function will take the smallest scalar whose reciprocal will put the point q in the set K.

Definition 3.3: Boundary.

Given a convex set K, the points $\boldsymbol{b} \in K$ such that for every $\varepsilon > 1$, $\varepsilon\boldsymbol{b} \notin K$, are the points on the boundary of K, $B(K)$.

You can also think of the boundary as points whose distance from the origin is 1, which is highlighted in the following proposition.

Proposition 3.4: Given the convex set K that defines the normed linear space $(\mathbb{R}^2, \| \cdot \|_K)$, K has the property that for elements that fall on the boundary of K, their norm equals 1.

Proof: Let $\boldsymbol{q} \in K$ and be on the boundary of the set K, meaning that going out any further from the origin past $\boldsymbol{q}$ would result in being outside of K. This means there are no elements of K strictly larger than $\boldsymbol{q}$. So in order for $\boldsymbol{q}/t \in K$ to be true, $t \geq 1$ must be a condition on t. Obviously $t = 1$ will work, and since the infimum function calls for the smallest t, $\| \boldsymbol{q} \|_K = 1$.

Since we see that norms can be defined by convex sets we ask ourselves what it takes to define a convex set. We could try listing each point one by one, but that won't be possible since we want to have an infinite number of points in any given convex set. The approach we want to take is to define a curve that separates the plane into two pieces so that the piece that contains the origin is a convex set.

This curve then becomes the boundary of the set it defines. This sounds like a really simple process but it relies on the Jordan Curve Theorem which uses topological concepts beyond the scope of this paper. We refer the reader to [2] for more information. Whenever we make reference to a curve we're implying that the curve is symmetrical about the origin, and the set it defines is convex, and therefore defines a norm, which in turn defines a metric. That's worth summarizing in the following theorem.

Theorem 3.5: The Hierarchy of Spaces.

Given a curve in $\mathbb{R}^2$ which defines a convex set K, let the normed linear space be $(\mathbb{R}^2, \|\cdot\|_K)$ by Definition 3.2 and the metric space be $(\mathbb{R}^2, d_K)$ by Proposition 1.4.

Theorem 3.5 just recaps a combination of facts that we've gathered throughout the paper with the punchline being a curve defines a metric. Keep in mind there are quite a few constraints on what that curve can be, but that hardly distracts from the theorem. Since we have this theorem we should start looking to define curves with the properties we are looking for.

Definition 3.6: Regular 2n-gon.

The vertices and connecting line segments of the $2n + 1$ points given by

$$\{V_{(k,n)}\} = \left\{\left(\cos\left(\pi\frac{k}{n}\right), \sin\left(\pi\frac{k}{n}\right)\right) | 0 \leq k \leq 2n; n \in \mathbb{N} - \{1\}; k \in \mathbb{Z}\right\}$$

Constructs a regular 2n-gon centered about the origin.

From the construction of these 2n-gons we have a curve defining a convex set and everything else we'd want to define comes from the hierarchy of spaces theorem. When we refer to the "2n-gon" we are formally defining it as this object described above as well as the norm $\|\cdot\|_{2n-gon}$ and metric

d_{2n-gon} it defines. We had to avoid polygons with an odd number of vertices since that shape wouldn't be symmetrical about the origin, despite the rotational and reflection symmetry.

Definition 3.7: Vector in $\mathbb{R}^2$.

Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^2$. The vector from $\boldsymbol{p}$ to $\boldsymbol{q}$ is $\boldsymbol{y} = \boldsymbol{q} - \boldsymbol{p}$.

Your reaction to this Definition 3.7 is probably "why in the world do we need to define that?". We have made this definition to emphasize one very small key detail; we think of vectors as starting at the origin. This is so taking norms of vectors requires no shifting around.

Definition 3.8: Regular 2n-gon segment vector.

The segment vector from $V_{(k+1,n)}$ to $V_{(k,n)}$ of the 2n-gon is denoted $\boldsymbol{s}_{(k,n)}$ for $0 \leq k \leq 2n-1$.

$$\boldsymbol{s}_{(k,n)} = V_{(k+1,n)} - V_{(k,n)}.$$

As an example $\boldsymbol{s}_{(6,13)}$ is the 6th segment vector of the regular 26-gon. To help keep track of the notation, remember that $\boldsymbol{s}$ stands for segment and is bold because it's defined like a vector. On the other hand V stands for vertex.

Corollary 3.9: $\| V_{(k,n)} \|_{2n-gon} = 1$

Proof: This comes as direct result of Proposition 3.4 and the fact that the vertices are part of the curve that defined the 2n-gon norm.

We write Corollary 3.9 only because it becomes useful later on. This next lemma is also written out for similar reasoning.

Lemma 3.10: If $\boldsymbol{y}$ is the vector from $\boldsymbol{p}$ to $\boldsymbol{q}$, then under any metric d_x defined from a norm$\| \cdot \|_x$, $\| \boldsymbol{y} \|_x = d_x[\boldsymbol{p}, \boldsymbol{q}]$.

Proof: We know that d_n is defined from our definition of metric from a norm, which is defined as $d_x[\boldsymbol{p}, \boldsymbol{q}] = \| \boldsymbol{q} - \boldsymbol{p} \|_x = \| \boldsymbol{y} \|_x$ from our definition of vector.

We want some way to say that two vectors in a metric are in the same direction. If fact, we're going to want to say that two vectors in two different metrics are in the same direction. Of course this means that the two metrics will have to be working under the same set, namely $\mathbb{R}^2$, so that the vectors are comparable. For this we will define an equivalent class on direction.

Definition 3.11: Equivalent direction of vectors.

Two vectors, $\boldsymbol{y_1}, \boldsymbol{y_2} \in \mathbb{R}^2$, are in an *equivalent direction* if and only if there exists a $m \in \mathbb{R}^+$ so that $\boldsymbol{y_1} = k\boldsymbol{y_2}$. Two vectors are *parallel* if and only if there exists $0 \neq l \in \mathbb{R}$ such that $\boldsymbol{y_1} = l\boldsymbol{y_2}$.

Not only are we giving a careful definition of direction, but we are separating the ideas of the distances from one point to another and the direction from one point to another. The real number $d_n[\boldsymbol{p}, \boldsymbol{q}]$ is the distance from $\boldsymbol{p}$ to $\boldsymbol{q}$ and vector $\boldsymbol{y} = \boldsymbol{q} - \boldsymbol{p}$ is the direction. We have shown that $\boldsymbol{y}$ also retains the distance when looked at under the norm from Lemma 3.10. This means we don't really lose any important information about $d_n[\boldsymbol{p}, \boldsymbol{q}]$ when we convert it to $\boldsymbol{y}$. This makes vectors a strictly superior object to use in many situations.

Definition 3.12: Unit Circle.

Let $(N, \|\cdot\|_x)$ be a normed linear space, then let $\Omega_{\mathrm{x}} = C(\mathbf{0}, 1) = \{\boldsymbol{y} \in N \mid d_x(\mathbf{0}, \boldsymbol{y}) = 1\} = \{\boldsymbol{y} \in N \mid \| \boldsymbol{y} \|_x = 1\}$, be the *unit circle* centered at the origin.

We can clearly see that the unit circle is the same at the boundary of the convex set since both sets include all points that have length one from the origin and no other points. We also showed 3 different unit circles in Figure 1, we just didn't refer to them that way because we hadn't defined it yet.

Definition 3.13: Curve Preserving.

Given unit circles Ω_{X} and Ω_{Y} we define $f: X \to Y$ to be a curve preserving map if $f(\boldsymbol{u}) \in \Omega_{\mathrm{Y}}$ for all $\boldsymbol{u} \in \Omega_{\mathrm{X}}$.

Since a curve defines a norm, it should be the case that a norm preserving map can be obtained by a curve preserving map. Below is such a proof.

Proposition 3.14: If f is a linear, curve preserving map from Ω_{X} to Ω_{Y} on $\mathbb{R}^2$ where X and Y are convex sets that define normed linear spaces $(\mathbb{R}^2, \|\cdot\|_X)$ and $(\mathbb{R}^2, \|\cdot\|_Y)$, then f is a norm preserving map.

Proof: We will show $\| f(\boldsymbol{x}) \|_X = \| \boldsymbol{x} \|_Y$ for all $\boldsymbol{x} \in \mathbb{R}^2$. If $\boldsymbol{x} = (0,0)$ then since f is linear $f(\boldsymbol{x}) = (0,0)$ and $\| (0,0) \|_X = 0 = \| (0,0) \|_Y$ by the properties of norms. For all other $\boldsymbol{x} \in \mathbb{R}^2$, $\| \boldsymbol{x} \|_X$ is a positive real number, so let $\| \boldsymbol{x} \|_X = k$. Again by the properties of norms

$$1 = \frac{\| \boldsymbol{x} \|_X}{k} = \left\| \frac{\boldsymbol{x}}{k} \right\|_X.$$

This means that $\frac{\boldsymbol{x}}{k} \in \Omega_{\mathrm{X}}$. Then since f is curve preserving

$$\left\| f\left(\frac{\boldsymbol{x}}{k}\right) \right\|_Y = 1.$$

Since f is linear we conclude

$$1 = \left\| f\left(\frac{\boldsymbol{x}}{k}\right) \right\|_Y = \left\| \frac{1}{k} f(\boldsymbol{x}) \right\|_Y = \frac{1}{k} \| f(\boldsymbol{x}) \|_Y,$$

$$\| f(\boldsymbol{x}) \|_Y = k = \| \boldsymbol{x} \|_X.$$

And so to summarize:

Theorem 3.15: The Hierarchy of Preservation.

Given f, a linear, origin fixing, curve preserving map from Ω_X to Ω_Y on $\mathbb{R}^2$, f is also linear, norm preserving map from $(\mathbb{R}^2, \| \cdot \|_X)$ to $(\mathbb{R}^2, \| \cdot \|_Y)$ and thus an isometry from $(\mathbb{R}^2, d_X)$ to $(\mathbb{R}^2, d_Y)$. The proof follows from Proposition 3.14 and Lemma 2.8.

This explains why the unit circles in Figure 1 looked so similar. Mapping one unit circle to to the other implies that the metrics are isometric. It seems obvious that you can map one unit circle to another when the picture looks nice. To keep pictures from influencing our judgment we're going to assume that the picture has no relevance to isometries unless we actually find the function that that does this mapping for us.

Now we'll change gears one last time for section 3 and introduce something completely new, the unit circle intersection. We'll find these possibly more important than isometries for this paper.

Definition 3.16: Unit Circle Intersection (UCI).

Given, $X_1 = (\mathbb{R}^2, d_1)$ and $X_2 = (\mathbb{R}^2, d_2)$ where Ω_1 and Ω_2 are the respective unit circles. Let $UCI = \Omega_1 \cap \Omega_2$ with $EUCI$ being the case where either X_1 or X_2 is E, the Euclidian metric space.

Finally let $2nEUCI$ be the case where X_1 and X_2 are E and the 2n-gon norm, for example $6EUCI$ is the intersection of the Euclidean unit circle and the hexagonal unit circle.

The power of the UCI is that it allows us to compare metric spaces that aren't isometric. For example there is no isometry from the taxi cab metric to the Euclidian metric, but there is a non-empty UCI between them. The purpose of the UCI becomes clear with the following proposition.

Proposition 3.17: Given different metric spaces, $X_1 = (\mathbb{R}^2, d_1)$ and $X_2 = (\mathbb{R}^2, d_2)$ with vectors $\boldsymbol{y}$, $\boldsymbol{x} \in \mathbb{R}^2$ such that $\boldsymbol{x}$ and $\boldsymbol{y}$ have equivalent directions, then $\| \boldsymbol{y} \|_1 = \| \boldsymbol{y} \|_2$ and $\| \boldsymbol{x} \|_1 = 1$ if and only if $\boldsymbol{x} \in UCI(X_1, X_2)$.

Proof: Let $\| \boldsymbol{y} \|_1 = \| \boldsymbol{y} \|_2$. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ share an equivalent direction, there exists a $k \in \mathbb{R}^+$ such that $k\boldsymbol{x} = \boldsymbol{y}$. So

$$\| \boldsymbol{y} \|_1 = \| \boldsymbol{y} \|_2$$

$$\| k\boldsymbol{x} \|_1 = \| k\boldsymbol{x} \|_2$$

$$|k|\| \boldsymbol{x} \|_1 = |k|\| \boldsymbol{x} \|_2$$

$$k\| \boldsymbol{x} \|_1 = k\| \boldsymbol{x} \|_2$$

$$\| \boldsymbol{x} \|_1 = \| \boldsymbol{x} \|_2.$$

Since $\| \boldsymbol{x} \|_1 = 1$, then$\| \boldsymbol{x} \|_2 = 1$. So $\boldsymbol{x} \in \Omega_1$ and $\boldsymbol{x} \in \Omega_2$, thus $\boldsymbol{x} \in UCI(X_1, X_2)$.

Let $\boldsymbol{x} \in UCI(X_1, X_2)$, then $\boldsymbol{x} \in \Omega_1$ and $\boldsymbol{x} \in \Omega_2$ so $\| \boldsymbol{x} \|_1 = 1 = \| \boldsymbol{x} \|_2$. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ share an equivalent direction, there exists a $k \in \mathbb{R}^+$ such that $\boldsymbol{x} = k\boldsymbol{y}$. So

$$\| \boldsymbol{x} \|_1 = \| \boldsymbol{x} \|_2$$

$$\| k\boldsymbol{y} \|_1 = \| k\boldsymbol{y} \|_2$$

$$|k|\| \boldsymbol{y} \|_1 = |k|\| \boldsymbol{y} \|_2$$

$$k\| \boldsymbol{y} \|_1 = k\| \boldsymbol{y} \|_2$$

$$\| \boldsymbol{y} \|_1 = \| \boldsymbol{y} \|_2.$$

What we really care about was the last half of the proof, which implies that if you can find a point in the UCI between two metric spaces, then any vector in an equivalent direction to that one point is automatically measured the same in both metrics. This is the last piece we need before forming our circumference argument, the main result of the paper.

Section 4: Circumference of 2n-gons in 2n-gon space

We have now finished what one would consider to be the background material of this paper and we will now move towards proving our main result of computing the circumference of the unit circle of the 2n-gon for all n. This will be referred to as the circumference argument. The circumference argument will allow us to show these circumferences approach the Euclidian circumference as n approaches ∞.

Definition 4.1: Circumference of the 2n-gon.

The circumference of the 2n-gon is the sum of the length of all its segments,

$$C_n = \sum_{k=0}^{2n-1} \| \boldsymbol{s}_{(k,n)} \|_{2n-gon}.$$

This is a rather limiting definition of circumference since it can't be applied to a boundary where any part of it is curved. That won't be a problem for our purposes since we are only interested in polygons. For now we begin by showing that segments of the 2n-gon have an equivalent direction to some particular element of the $2nEUCI$.

Proposition 4.2: Let $n = 2z + 1$, $z \in \mathbb{N}$. Then the z^{th} segment vector, $\boldsymbol{s}_{(z,n)}$, of the 2n-gon has equivalent direction to the vector $V_{(0,n)}$, where $V_{(0,n)} = (1,0) \in 2nEUCI$.

Proof: Since $n = 2z + 1$, then $n - z = z + 1$. Using the definition of segment vector we see $\boldsymbol{s}_{(z,n)} = V_{(z,n)} - V_{(z+1,n)} = \left(\cos\left(\pi\frac{z}{n}\right), \sin\left(\pi\frac{z}{n}\right)\right) - \left(\cos\left(\pi\frac{z+1}{n}\right), \sin\left(\pi\frac{z+1}{n}\right)\right)$. First, we will focus on the y-coordinate. We will be using the trig identity $\sin(\theta) = \sin(\pi - \theta)$ in the first equality.

$$\sin\left(\pi\frac{z}{n}\right) - \sin\left(\pi\frac{z+1}{n}\right)$$

$$= \sin\left(\pi - \pi\frac{z}{n}\right) - \sin\left(\pi\frac{z+1}{n}\right)$$

$$= \sin\left(\pi\frac{n-z}{n}\right) - \sin\left(\pi\frac{z+1}{n}\right)$$

$$= \sin\left(\pi\frac{z+1}{n}\right) - \sin\left(\pi\frac{z+1}{n}\right)$$

$$= 0.$$

At this point it doesn't matter what the x-coordinate is, so let $\boldsymbol{s}_{(z,n)} = (a, 0)$. However it is in our interests to show that a is positive, which means we want to show

$$\cos\left(\pi\frac{z}{n}\right) - \cos\left(\pi\frac{z+1}{n}\right) > 0,$$

which is equivalent to

$$\cos\left(\pi\frac{z}{n}\right) > \cos\left(\pi\frac{z+1}{n}\right).$$

It's known that $\cos(\theta)$ is decreasing on the interval $(0, \pi)$, so it's sufficient to show that

$$0 < \frac{z}{n} < \frac{z+1}{n} < 1.$$

This is obviously true as $\frac{2z+1}{n} = 1$ and z and n are both positive. Then we claim that $V_{(0,n)}$ is in an equivalent direction to $(a, 0)$. This means we want to find an $m \in \mathbb{R}^+$ so that $mV_{(0,n)} = \boldsymbol{s}_{(z,n)}$.

$$V_{(0,n)} = \left(\cos\left(\pi\frac{0}{n}\right), \sin\left(\pi\frac{0}{n}\right)\right) = (\cos(0), \sin(0)) = (1,0)$$

In which case we let $m = a$ since a is positive so $mV_{(0,n)} = a(1,0) = (a,0) = \boldsymbol{s}_{(z,n)}$. Obviously $V_{(0,n)} \in 2nEUCI$ since $\| V_{(0,n)} \|_{2n-gon} = 1$ and $\| (1,0) \|_E = 1$.

Now we know that for odd n we can measure one segment of the 2n-gon in the Euclidian metric. This may seem insignificant but it's actually the most crucial step, as we now just have to show that all segments behave similarly. First we will find the length of this segment.

Lemma 4.3: The length $\| \boldsymbol{s}_{(z,n)} \|_{2n-gon} = 2\sin\left(\frac{\pi}{2n}\right)$.

Proof: From Proposition 4.2 we proved that $\boldsymbol{s}_{(z,n)}$ is in the equivalent direction of an element of $2nEUCI$ where n is odd. From Proposition 3.17 we see that $\| \boldsymbol{s}_{(z,n)} \|_{2n-gon} = \| \boldsymbol{s}_{(z,n)} \|_E$. Since we know more about the Euclidian metric we can use what we know there, particularly high school level geometry, to find the measurement. If we think of $\boldsymbol{s}_{(z,n)}$ as a line segment then we can draw a line from the midpoint of $\boldsymbol{s}_{(z,n)}$ to the origin. This line will be (Euclidean) perpendicular to $\boldsymbol{s}_{(z,n)}$ due to the symmetry of regular 2n-gons. We can then draw a line from either of $\boldsymbol{s}_{(z,n)}$ endpoints to the origin, creating a right triangle with the origin, $\boldsymbol{s}_{(z,n)}$ midpoint M, and $\boldsymbol{s}_{(z,n)}$ endpoint, see Figure 2 below. Then the endpoint is identified as either $V_{(z,n)}$ or $V_{(z+1,n)}$, both of which have length 1 from the origin in the Euclidian metric by definition. We can also find the interior angle at the origin to be $\frac{1}{2}\frac{2\pi}{2n} = \frac{\pi}{2n}$ from the symmetry of the regular 2n-gon. This makes the opposite side, $MV_{(z,n)}$, length $\sin\left(\frac{\pi}{2n}\right)$, which is half of the segment $\boldsymbol{s}_{(z,n)}$, and thus $\| \boldsymbol{s}_{(z,n)} \|_E = 2\sin\left(\frac{\pi}{2n}\right)$. Therefore $\| \boldsymbol{s}_{(z,n)} \|_{2n-gon} = 2\sin\left(\frac{\pi}{2n}\right)$.

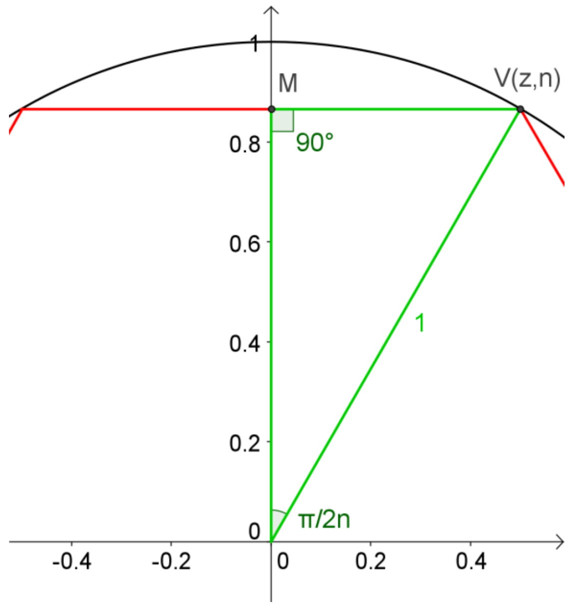

Figure 2

Now we have found the length of one segment we would like to say that each other segment is the same length. This would make sense since picking any segment of a regular polygon should be arbitrary and so result in the same side length. We can do better than that argument, as seem in the lemma below.

Lemma 4.4: Let r be a rotation by any integer multiple of $\frac{\pi}{n}$, then $r: (\mathbb{R}^2, d_{2n-gon}) \to (\mathbb{R}^2, d_{2n-gon})$ is an isometry.

Proof: A rotation by the Euclidian angle $\frac{\pi}{n}$ can be represented by the rotation matrix,

$$R = \begin{bmatrix} \cos \pi/n & -\sin \pi/n \\ \sin \pi/n & \cos \pi/n \end{bmatrix}.$$

If we apply this matrix to an arbitrary vertex of the 2n-gon, $V_{(k,n)} = \left(\cos\left(\pi \frac{k}{n}\right), \sin\left(\pi \frac{k}{n}\right)\right)$, we find

$$R * V_{(k,n)}$$

$$= \left[\cos\left(\pi\frac{k}{n}\right)\cos\left(\frac{\pi}{n}\right) - \sin\left(\pi\frac{k}{n}\right)\sin\left(\frac{\pi}{n}\right), \cos\left(\pi\frac{k}{n}\right)\sin\left(\frac{\pi}{n}\right) + \sin\left(\pi\frac{k}{n}\right)\cos\left(\frac{\pi}{n}\right)\right]$$

$$= \left[\cos\left(\pi\frac{k}{n} + \frac{\pi}{n}\right), \sin\left(\pi\frac{k}{n} + \frac{\pi}{n}\right)\right]$$

$$= \left[\cos\left(\pi\frac{k+1}{n}\right), \sin\left(\pi\frac{k+1}{n}\right)\right]$$

$$= V_{(k+1,n)}.$$

The second equality used two versions of the angle sum identity. We have just proved that this rotation takes each vertex to the sequentially next vertex. Since each segment of the 2n-gon is defined by the vertices each segment has been translated to the next as well. This shows the $\frac{\pi}{n}$ rotation is curve preserving, and so by the hierarchy theorem for preservation, the $\frac{\pi}{n}$ rotation is an isometry. Since $Isom(X)$ is a group under function composition by Theorem 2.4, all integer multiples of the $\frac{\pi}{n}$ rotation are also isometries.

We now have all the pieces in place to make what will became half of the circumference argument.

Theorem 4.5: Let $n = 2z + 1$, $z \in \mathbb{N}$. The circumference $C_n = 4n\sin\left(\frac{\pi}{2n}\right)$ and $\lim_n C_n = 2\pi$.

Proof: From Lemma 4.4, we see that we can rotate the $\boldsymbol{s}_{(z,n)}$ segment using an isometry to any other segment of the 2n-gon, so all segments of the 2n-gon must have the same length. Therefore by Lemma 4.3 the circumference of the 2n-gon is $4n\sin\left(\frac{\pi}{2n}\right)$ as there are $2n$ segments.

Using L'Hopital's rule we see

$$\lim_{n\to\infty} C_n$$

$$= \lim_{n\to\infty} 4n\sin\left(\frac{\pi}{2n}\right)$$

$$= 4 \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{2n}\right)}{n^{-1}}$$

$$= 4 \lim_{n\to\infty} \frac{(-1)\left(\frac{\pi}{2}\right)\mathrm{n}^{-2}\cos\left(\frac{\pi}{2n}\right)}{(-1)n^{-2}}$$

$$= 4\frac{\pi}{2}\lim_{n\to\infty}\cos\left(\frac{\pi}{2n}\right)$$

$$= 2\pi.$$

Our goal now is to show this also happens for 2n-gons when n is even. This won't be as easy because the edges of an even 2n-gon are not in an equivalent direction to an element of 2nEUCI. So to move forward we must modify the 2n-gon so that we do get such an equivalent direction.

Section 5: Circumference of 2m-goms in 2m-gom space

This section is much like section 4 in that we find a way to draw polygons over the origin and compute their circumference. This time we are going to address the even case that brings a similar and interestingly different challenge by considering the midpoint.

Definition 5.1: Modified 2n-gon, the 2m-gom.

The points and connecting line segments of the $2n + 1$ points given by

$$\{W_{(k,m)}\} = \left\{\frac{1}{\cos(\pi/2m)}\left(\cos\left(\pi\frac{k}{m} - \frac{\pi}{2m}\right), \sin\left(\pi\frac{k}{m} - \frac{\pi}{2m}\right)\right) | 0 \le k \le 2m; m \in \mathbb{N} - \{1\}; k \in \mathbb{Z}\right\}$$

constructs a regular 2m-gom centered over the origin.

Now that we've made a whole new set of regular 2n-gons, the 2m-goms, we're going to have to re-examine which vectors are elements of the set $2mEUCI$. First let's get some notation down for this new group of regular polygons centered about the origin. What were $V_{(k,n)}$, the vertices, of the 2n-gon are now $W_{(k,m)}$, as demonstrated by the 2m-gom definition. The segment vector $\boldsymbol{s}_{(k,n)}$, will now be $\boldsymbol{t}_{(k,m)}$ for the 2m-gom. It should be obvious what the norm $\|\cdot\|_{2m-gom}$ and metric d_{2m-gom} stand for. This time it won't be vertices of the 2m-gom that are elements of the $2mEUCI$ like was true for $2nEUCI$. Instead it will be the midpoints of the segments that are the elements of the $2mEUCI$. To show that every midpoint is a $2mEUCI$ the first step is showing just one midpoint is such an element.

Definition 5.2: Midpoint of a segment.

The midpoint of a given segment that starts and ends at the points $\boldsymbol{p}$ and $\boldsymbol{q}$ is $\frac{1}{2}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}$.

Lemma 5.3: For any 2m-gom the segment midpoint of $W_{(0,m)}$ and $W_{(1,m)}$ is the point $(1,0) \in 2mEUCI$.

Proof: We will show that $\frac{1}{2}W_{(0,m)} + \frac{1}{2}W_{(1,m)} = (1,0)$. To save writing let $a = \frac{1}{\cos(\pi/2m)}$. Let's first focus on the y-coordinate. Using $\sin(\theta) = -\sin(-\theta)$ we see

$$\frac{1}{2}a\left[\sin\left(0 - \frac{\pi}{2m}\right) + \sin\left(\frac{\pi}{m} - \frac{\pi}{2m}\right)\right]$$

$$= \frac{1}{2}a\left[\sin\left(-\frac{\pi}{2m}\right) + \sin\left(\frac{\pi}{2m}\right)\right]$$

$$= \frac{1}{2}a[0]$$

$$= 0.$$

Then using $\cos(\theta) = \cos(-\theta)$ we see

$$\frac{1}{2}a\left[\cos\left(0-\frac{\pi}{2m}\right)+\cos\left(\frac{\pi}{m}-\frac{\pi}{2m}\right)\right]$$

$$=\frac{1}{2}a\left[\cos\left(-\frac{\pi}{2m}\right)+\cos\left(\frac{\pi}{2m}\right)\right]$$

$$=\frac{1}{2}\left(\frac{1}{\cos(\pi/2m)}\right)\left[2\cos\left(\frac{\pi}{2m}\right)\right]$$

$$=1.$$

Thus $\frac{1}{2}W_{(0,m)}+\frac{1}{2}W_{(1,m)}=(1,0)$, which means $\|(1,0)\|_{2m-gom}=1$ since it's on the 2m-gom by Proposition 3.4. Then obviously as $\|(1,0)\|_E=1$, $(1,0)\in 2mEUCI$.

We have shown that one midpoint is an element of the set $2mEUCI$. This is the majority of the work in showing all midpoints are elements of $2mEUCI$.

Proposition 5.4: Given f is a linear isometry of both X_1, $f : X_1 \to X_1$ and X_2, $f : X_2 \to X_2$, then $f(UCI(X_1,X_2)) = UCI(X_1,X_2)$.

Proof: Take an arbitrary $\boldsymbol{x}\in UCI(X_1,X_2)$. Then $\| f(\boldsymbol{x}) \|_1 = \| \boldsymbol{x} \|_1 = 1 = \| \boldsymbol{x} \|_2 = \| f(\boldsymbol{x}) \|_2$. This shows $f(UCI(X_1,X_2)) \subseteq UCI(X_1,X_2)$. Since $Isom(X)$ is a group by Theorem 2.4, f^{-1} is an isometry of both X_1, $f^{-1} : X_1 \to X_1$ and X_2, $f^{-1} : X_2 \to X_2$. Thus $f^{-1}(UCI(X_1,X_2)) \subseteq UCI(X_1,X_2)$. Then

$$UCI(X_1,X_2) = f\left(f^{-1}\big(UCI(X_1,X_2)\big)\right) \subseteq f\big(UCI(X_1,X_2)\big).$$

Therefore $f(UCI(X_1,X_2)) = UCI(X_1,X_2)$.

Then by putting together the past few statements we can prove the following.

Corollary 5.5: All midpoints of the 2m-gom segment vectors are elements of $2mEUCI$.

Proof: From Lemma 5.3 we know that one midpoint is an element of the $2mEUCI$ and from Proposition 5.4 any isometry of both $(\mathbb{R}^2, d_{\mathrm{2m-gom}})$ and Euclidian metric spaces will take elements of $2mEUCI$ to other elements of $2mEUCI$. Any rotation is an isometry of the Euclidian metric, and as shown in Lemma 4.4 with trivial changes to the proof, we can show any rotation multiple of $\frac{\pi}{m}$ is an isometry of any 2m-gom. Thus every midpoint must be an element of the $2mEUCI$.

Now that we know the elements of $2mEUCI$ we can move forward with finishing the circumference argument. Similarly to before we start by showing that a single segment is in an equivalent direction to an element of the $2mEUCI$.

Proposition 5.6: Let $m = 2z$, $z \in \mathbb{N}$. There z^{th} segment vector of the 2m-gom, $\boldsymbol{t}_{(z,m)}$, has equivalent direction to the vector $(1,0) \in 2mEUCI$.

Proof: Since $m = 2z$, $m - z = z$. Again we let $a = \frac{1}{\cos(\pi/2m)}$. From the definition of segment vector we see

$$\boldsymbol{t}_{(z,m)} = W_{(z+1,m)} - W_{(z,m)}$$

$$= a\left(\cos\left(\pi\frac{z+1}{m} - \frac{\pi}{2m}\right), \sin\left(\pi\frac{z+1}{m} - \frac{\pi}{2m}\right)\right) - a\left(\cos\left(\pi\frac{z}{m} - \frac{\pi}{2m}\right), \sin\left(\pi\frac{z}{m} - \frac{\pi}{2m}\right)\right).$$

As in the 2n-gon case, we will look at the y-coordinate first and use the trig-identity $\sin(\theta) = \sin(\pi - \theta)$.

$$a\left[\sin\left(\pi\frac{z+1}{m} - \frac{\pi}{2m}\right) - \sin\left(\pi\frac{z}{m} - \frac{\pi}{2m}\right)\right]$$

$$= a\left[\sin\left(\pi - \pi\frac{z+1}{m} + \frac{\pi}{2m}\right) - \sin\left(\pi\frac{z}{m} - \frac{\pi}{2m}\right)\right]$$

$$= a\left[\sin\left(\pi\frac{2m-2(z+1)+1}{2m}\right)-\sin\left(\pi\frac{2z-1}{2m}\right)\right]$$

$$= a\left[\sin\left(\pi\frac{2(m-z)-1}{2m}\right)-\sin\left(\pi\frac{2(m-z)-1}{2m}\right)\right]$$

$$= 0.$$

From this let $\boldsymbol{t}_{(z,m)} = (b, 0)$. We see that it doesn't really matter what the x-coordinate is as long as it is positive. Notice $a = \frac{1}{\cos(\pi/2m)} > 0$ for $m > 1$ which accounts for all possible 2m-goms. Recall from the Definition 5.1, $m \geq 2$. We will prove

$$0 < \ a\left[\cos\left(\pi\frac{z+1}{m}-\frac{\pi}{2m}\right)-\cos\left(\pi\frac{z}{m}-\frac{\pi}{2m}\right)\right].$$

Notice that the following inequalities are equivalent.

$$a\cos\left(\pi\frac{z}{m}-\frac{\pi}{2m}\right) < a\cos\left(\pi\frac{z+1}{m}-\frac{\pi}{2m}\right)$$

$$\cos\left(\pi\frac{2z-1}{2m}\right) < \cos\left(\pi\frac{2z+1}{2m}\right)$$

$$\cos\left(\pi\frac{2z-1}{4z}\right) < \cos\left(\pi\frac{2z+1}{4z}\right).$$

We know that cosine is a decreasing function on the interval $(0, \pi)$ so it's sufficient to show

$$0 < \frac{2z-1}{4z} < \frac{2z+1}{4z} < 1.$$

This is true for $z > 1/2$ which accounts for all possible z as $z \geq 1$. Now we can find an $c \in \mathbb{R}^+$ so that $c(1,0) = \boldsymbol{t}_{(z,m)}$ by letting $c = b$. Thus $\boldsymbol{t}_{(z,m)}$ and $(1,0)$ are in an equivalent direction.

Using the blueprint laid out by section 4 we continue to finding the length of this one individual side, the z^{th} side.

Proposition 5.7: The length $\| \boldsymbol{t}_{(z,m)} \|_{2m-gom} = 2\tan\left(\frac{\pi}{2m}\right)$.

Proof: From Proposition 5.6 we proved $\boldsymbol{t}_{(z,m)}$ is in the equivalent direction of an element of $2mEUCI$ where m is even. From Proposition 3.17 we see that $\| \boldsymbol{t}_{(z,m)} \|_{2m-gom} = \| \boldsymbol{t}_{(z,m)} \|_{E}$. Again we know more about the Euclidian metric so we will use that metric to find the measurement. If we think of $\boldsymbol{t}_{(z,m)}$ as a line segment then we can draw a line from the midpoint of $\boldsymbol{t}_{(z,m)}$ to the origin. This line will be (Euclidean) perpendicular to $\boldsymbol{t}_{(z,m)}$ due to the symmetry of regular 2m-goms. We can then draw a line from either of $\boldsymbol{t}_{(z,m)}$ endpoints to the origin, creating a right triangle with the origin, $\boldsymbol{t}_{(z,m)}$ midpoint M, and a $\boldsymbol{t}_{(z,m)}$ endpoint, $W_{(z,m)}$ or $W_{(z+1,m)}$. See Figure 3 below. Then the midpoint we know has length 1 from origin in the Euclidian metric by Corollary 5.5. We can also find the interior angle at the origin to be $\frac{1}{2}\frac{2\pi}{2m} = \frac{\pi}{2m}$ from the symmetry of the regular 2m-gom. This makes the opposite side, $MW_{(z,m)}$, length $\tan\left(\frac{\pi}{2m}\right)$, which is half of the segment $\boldsymbol{t}_{(z,m)}$, and thus $\| \boldsymbol{t}_{(z,m)} \|_{E} = 2\tan\left(\frac{\pi}{2m}\right)$. Thus $\| \boldsymbol{t}_{(z,m)} \|_{2m-gom} = 2\tan\left(\frac{\pi}{2m}\right)$.

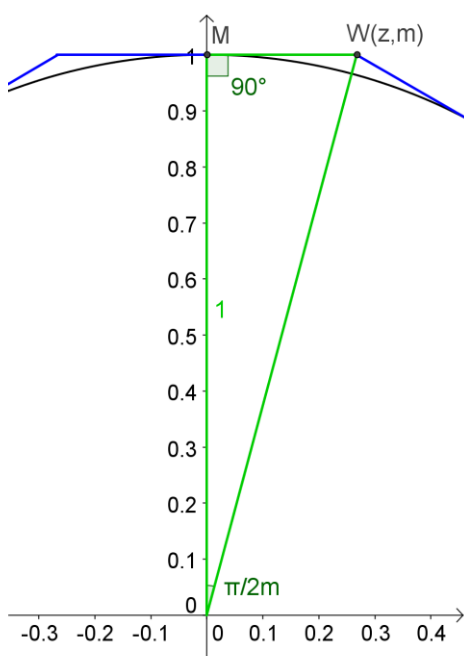

Figure 3

And then the final step of this process concludes with the following theorem.

Theorem 5.8: Let $m = 2z$, $z \in \mathbb{N}$. The circumference $C_m = 4m\tan\left(\frac{\pi}{2m}\right)$ and $\lim_m C_m = 2\pi$.

Proof: Similarly to the Theorem 4.5, the rotations of the 2m-goms are isometries by Lemma 4.4 so the length of each segment is the same. Therefore $C_m = 4m\tan\left(\frac{\pi}{2m}\right)$ as there are 2m segments of length $2\tan\left(\frac{\pi}{2m}\right)$.

Using L'Hopital's rule we see

$$\lim_{m\to\infty} C_m$$

$$= \lim_{m\to\infty} 4m\tan\left(\frac{\pi}{2m}\right)$$

$$= 4\lim_{m\to\infty} \frac{\tan\left(\frac{\pi}{2m}\right)}{m^{-1}}$$

$$= 4\lim_{m\to\infty} \frac{(-1)\frac{\pi}{2}m^{-2}\sec^2\left(\frac{\pi}{2m}\right)}{(-1)m^{-2}}$$

$$= 4\frac{\pi}{2}\lim_{m\to\infty} \sec^2\left(\frac{\pi}{2m}\right)$$

$$= 2\pi.$$

And so we see both the odd and even case approach 2π and n and m go towards infinity respectively. This result isn't surprising if we imagine that as a regular polygon approaches infinitely many sides the shape approaches a circle with a circumference of 2π. The interesting part is that we showed this happens when you inspect the circumference under a sequence of metrics defined by regular polygons, not under the Euclidian metric.

Section 6: The Circumference Argument

In this section we show that the two polygon constructions, 2n-gons and 2m-goms, define isometric metric spaces. From this we show the circumferences are the same and conclude by investigating the notion of pi in these metric spaces.

Proposition 6.1: When $m = n$ there exists an isometry between the 2n-gon and the 2m-gom metrics that maps vertex $V_{(k,n)}$ to vertex $W_{(k,m)}$.

Proof: From the hierarchy theorem for preservation we know that a curve preserving map from the 2n-gon to the 2m-gom will be an isometry. Pick an arbitrary 2n-gon and an arbitrary vertex, $V_{(k,n)}$, of that 2n-gon. We claim f given by

$$f(\boldsymbol{x}) = \frac{1}{\cos(\pi/2n)} \begin{bmatrix} \cos(-\pi/2n) & -\sin(-\pi/2n) \\ \sin(-\pi/2n) & \cos(-\pi/2n) \end{bmatrix} \boldsymbol{x}$$

will have the property $f\left(V_{(k,n)}\right) = W_{(k,m)}$ and thus be curve preserving. Again let $a = \frac{1}{\cos(\pi/2n)}$. By Definition 3.6 $n \geq 2 > 1$ so $a > 0$. Then by letting $n = m$,

$$f\left(V_{(k,n)}\right)$$

$$= a \begin{bmatrix} \cos(-\pi/2n) & -\sin(-\pi/2n) \\ \sin(-\pi/2n) & \cos(-\pi/2n) \end{bmatrix} V_{(k,n)}$$

$$= a \left[\cos\left(\pi\frac{k}{n}\right)\cos\left(\frac{-\pi}{2n}\right) - \sin\left(\pi\frac{k}{n}\right)\sin\left(\frac{-\pi}{2n}\right), \cos\left(\pi\frac{k}{n}\right)\sin\left(\frac{-\pi}{2n}\right) + \sin\left(\pi\frac{k}{n}\right)\cos\left(\frac{-\pi}{2n}\right)\right]$$

$$= a \left[\cos\left(\pi\frac{k}{n} - \frac{\pi}{2n}\right), \sin\left(\pi\frac{k}{n} - \frac{\pi}{2n}\right)\right]$$

$$= \frac{1}{\cos(\pi/2m)}\left(\cos\left(\pi\frac{k}{m} - \frac{\pi}{2m}\right), \sin\left(\pi\frac{k}{m} - \frac{\pi}{2m}\right)\right)$$

$$= W_{(k,m)}.$$

Since f sends the k^{th} vertex of the 2n-gon to the k^{th} vertex of the 2m-gom and the curves of both sets are defined by the segments between vertices, f is a curve preserving map. Also, since rotations and multiplying by a non-zero scalar are both linear origin fixing operators, f is linear and origin preserving. Therefore f is an isometry between the 2n-gon and the 2m-gom metrics by Theorem 3.15.

And now the circumference argument emerges.

Theorem 6.2: The circumference of the 2n-gon, $C_n = C_m$, the circumference of the 2m-gom.

Proof: Since there exists a linear isometry, f, between the 2n-gon and the 2m-gom that sends vertices to vertices, and the segments are defined by the vertices, then the isometry sends segments to segments. Thus $\| \boldsymbol{s}_{(k,n)} \|_{2n-gon} = \| f(\boldsymbol{s}_{(k,n)}) \|_{2m-gom} = \| \boldsymbol{t}_{(k,m)} \|_{2m-gom}$ for all $0 \leq k \leq 2n-1 = 2m-1$. Thus

$$\sum_{k=0}^{2n-1} \| \boldsymbol{s}_{(k,n)} \|_{2n-gon} = \sum_{k=0}^{2m-1} \| \boldsymbol{t}_{(k,m)} \|_{2m-gom}$$

$$C_n = C_m$$

This result may seem like a contradiction to earlier results that found $C_n = 4n\sin\left(\frac{\pi}{2n}\right)$ and $C_m = 4m\tan\left(\frac{\pi}{2m}\right)$. That would be misinterpretation of those earlier results since they are defined for n odd ≥ 3 and m even ≥ 2 so there is no overlap. However, we get a complete set of the circumferences C_n, of every 2n-gon with 4 or more vertices in the metric space defined by that 2n-gon. We could then look at C_n as a sequence of numbers with the first term being C_2, corresponding to the circumference of the circle of radius 1 under the 4-gon (square) metric.

Definition 6.3: The Circumference Sequence.

For $n \geq 2, n \in \mathbb{N}$,

$$C_n = \begin{cases} 4n \sin\left(\frac{\pi}{2n}\right) & \text{for } n \text{ odd} \\ 4n \tan\left(\frac{\pi}{2n}\right) & \text{for } n \text{ even} \end{cases}$$

Pi we think of as the ratio between circumference and diameter. Since the diameter of any unit circle is always 2 we can define a sequence of π_n to be such a ratio.

Definition 6.4: The Pi Sequence.

For $n \geq 2, n \in \mathbb{N}$,

$$\pi_n = \begin{cases} 2n \sin\left(\frac{\pi}{2n}\right) & \text{for } n \text{ odd} \\ 2n \tan\left(\frac{\pi}{2n}\right) & \text{for } n \text{ even} \end{cases}$$

Obviously $\pi_n = \frac{1}{2} C_n$, so $\lim_n \pi_n = \frac{1}{2} \lim_n C_n = \pi$ which is also expected. One might want to know how fast this convergence happens. Below is a table that gives both cases of what pi is a selection of 2n-gon spaces. The bold values indicate the applicable one for that 2n-gon. For comparison the approximate value of Euclidian pi is 3.14159265358979.

$2n$	$2n\sin\left(\frac{\pi}{2n}\right)$	$2n\tan\left(\frac{\pi}{2n}\right)$
8	3.06146746	**3.31370850**
10	**3.09016994**	3.24919696
50	**3.13952598**	3.14473336
100	3.14107591	**3.14262660**
1000	3.14158749	**3.14160299**
10000	3.14159260	**3.14159276**

We see that for the case where n is odd pi is always underestimated by the middle column and when n is even pi is overestimated by the right column. This comes from the fact that the odd case, coming from Definition 3.6, the 2n-gon, is making a polygon inscribed into the Euclidian unit circle. Since it inscribes the circle it's always shortcutting the curvature of the Euclidian circle. For the even case that follows from Definition 5.1, the 2m-gom, the polygon constructed is circumscribing the Euclidian circle. This makes the Euclidian circumference shorter than the 2m-gom and thus these cases always overestimate pi.

Section 7: Conclusion and References

We started from the bottom, defining metric and norm, and worked our way up to defining non-conventional terms such as UCI and 2m-gom, to give a result on countably infinite distinct metrics. In section one we discussed and explored the differences between metric and norm and the ways in which you might get one from the other. In section 2 we introduced the isometry, a useful way to compare different metrics. By showing that norm preserving was just as good we could make our proofs easier to manage as exemplified in Theorem 2.9.

In section 3 we introduced convex sets that we saw worked at a level even more elementary than norms. This also allowed us to define 2n-gons and 2m-goms which became useful for generalizing how these metrics behaved. It also enforced the idea that there were two distinct cases for how a convex set shaped as a polygon would behave. In section 4 it all came together, referencing back to earlier sections often and building to the final result. For a result so obvious and reasonable one would expect the work leading up to it to be much shorter. It speaks well for the complexity of geometry and how nothing is ever guaranteed. We also saw quite a few uses of other areas of mathematics used in our proof, including abstract algebra for showing $Isom(X)$ was a group and calculus for L'Hopital's rule.

With more time we would have explored the asymmetric spaces defined by polygons such as the triangle and pentagon. They were excluded from this result as they did not fit the symmetric property for construction of a norm from a convex set. We had been preparing to take on this topic by being very careful in the way we defined direction in Definition 3.11, even though the symmetric property would have made parallel good enough for all of our proofs. Although we completed our goal of computing circumference there is much yet to be explored in this area of mathematics.

References:

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